Answers:

1. A
2. B
3. C
4. B
5. D
6. A
7. E
8. A
9. C
10. A
11. D
12. B
13. B
14. C
15. D
16. A
17. D
18. B
19. C
20. C
21. A
22. D
23. B
24. B
25. A
26. C
27. E
28. D
29. B
30. C

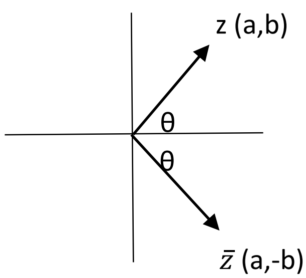
Solutions:

1. $\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & -4 \end{bmatrix}$.

2. $2x_1 - 4x_2 + 3x_3 = k$
   $2(1) - 4(2) + 3(3) = k$
   $k = 3$

3. $1x_1 + \frac{1}{2!}x_2 + \frac{1}{3!}x_3 + \ldots = k$
   $1(2) + \frac{1}{2!}(2) + \frac{1}{3!}(2) + \ldots = k$
   $2(e-1) = k$
   $\ln(k) = \ln(2(e-1)) = \ln(e-1) + \ln(2)$

4.



angle = $2\tan^{-1}(\frac{b}{a})$

5. $\begin{vmatrix} 1 \\ -2 \\ 5 \end{vmatrix} = a\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} + b\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} + c\begin{vmatrix} 2 \\ -1 \\ 1 \end{vmatrix}$

Solving this yields a system of equations with a unique solution: $a = -6, b = 3, c = 2$.
Then, $\frac{a}{b \cdot c}$ = -1.

6. Proj(u,v) = $\left(\frac{u \cdot v}{\|v\|^2}\right) v$ where $\left(\frac{u \cdot v}{\|v\|^2}\right)$ determines the magnitude and v the direction
   = $(\frac{57}{74}, \frac{38}{37}, \frac{133}{74})$.

7. All of the statements are true about traces, making E the correct answer.

8. In two dimensions, every rotation matrix has the following form: $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for some θ and det($R(\theta)$) = $cos^2\theta + sin^2\theta = 1$.

9. $\begin{bmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -2 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 7 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix}$ by replacing Row_1 with $-2Row_2+Row_1$, Row_3 with $3Row_2+Row_3$, and Row_4 with Row_2+Row_4. Note this does not change the determinant of the matrix. Now, expansion by minors gives a determinant of 103.

10. Volume of a parallelepiped:

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = |-2| = 2.$$ So the volume of a tetrahedron is 1/6 of this or 1/3.

11. Any 2 DISTINCT vectors form a single, unique plane. Choices III and IV are not distinct vectors, making the answer I and II only.

12. Reduce the matrix to row-echelon via elementary row operations:

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ So the rank = 2.

13. In general, $p(t) = \det(A-tI_n)(-1)^n$

→ $tr(M) = 9$; $\det(M) = 20$; so $p(t) = t^2-9t+20$

14. $\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \rightarrow \lambda = 3, 5$

15. Using Cramer's:

$$x = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ 5 & 6 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 2 & 6 & 13 \end{vmatrix}} = -6 \; ; \; y = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 6 \\ 2 & 5 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 2 & 6 & 13 \end{vmatrix}} = 5; \text{ and } z = x = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 2 & 6 & 13 \end{vmatrix}} = -1.$$ Thus, xyz = 30.

16. $M^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$. So, $tr(M^{-1}) = -1 + 1 + 8 = 8$

17. Orthogonal vectors have a dot product of 0. U_4 is the only vector not orthogonal to the three others.

18. Use Cramer's to solve:

$$x = \frac{\begin{vmatrix} a^{\frac{5}{2}} & -\sqrt{b} \\ b^{\frac{5}{2}} & \sqrt{a} \end{vmatrix}}{\begin{vmatrix} \sqrt{a} & -\sqrt{b} \\ \sqrt{b} & \sqrt{a} \end{vmatrix}} = \frac{a^3+b^3}{a+b} = a^2 - ab + b^2$$

19. $AA = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$A^2A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} = 2\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$A^3 = 2A^2$

$A^4 = 4A^2$

⋮

$A^n = 2^{n-2}A^2$

20. There are 10^9 combinations possible, but only 10^6 are symmetric. So, the probability that Vishal will win is $\frac{10^6}{10^9} = 1/1000$.

21. For two vectors to be perpendicular, their dot product must be 0. Answer A only results in this.

22. Trace = the sum of the elements on the main diagonal. For an n x n identity matrix, the sum is n.

23. The matrix satisfies the definition of skew-symmetric, where $A = -A^T$.

24. $|\det(B^{-1}A^{-1})| = |\det(B^{-1})\det(A^{-1})| = \left|\frac{1}{\det(B)\det(A)}\right|$. Here det(A) = 2 and det(B) = -2. Thus, $|\det(B^{-1}A^{-1})| = \left|\frac{1}{(-2)(2)}\right| = \frac{1}{4}$.

25. The matrix has the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{1}{2}$. Thus, $\theta = 210°$.

26. We know $a \cdot b = \|a\|\|b\| \cos\theta$ and $\|axb\| = \|a\|\|b\| \sin\theta$. Thus $\frac{\|axb\|}{a \cdot b} = \frac{\|a\|\|b\| \sin\theta}{\|a\|\|b\| \cos\theta} = \tan\theta = \sqrt{3}$. Thus, θ = 60°, and the sum of the digits are 6.

27. Here, we are multiplying a 4x2 and a 3x3, which is not possible by the rules of matrix multiplication, hence our answer is E) NOTA.

28. Eigenvalues are given by det(A-λI) = 0. Hence, $\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0.$

→ $(3-\lambda)(3-\lambda) - 1 = 0$
→ $\lambda^2 - 6\lambda + 8 = 0$
→ $(\lambda - 2)(\lambda - 4) = 0$
→ $\lambda = 2, 4$

Now, we are looking for solutions (x, y) to the equation $\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Substituting in these two values of :

$\lambda = 2$ → $\left.\begin{matrix} x - y = 0 \\ -x + y = 0 \end{matrix}\right\}$ → $x = y$. Thus, $\lambda = 2$ has eigenvectors of $\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda = 4$ → $\left.\begin{matrix} -x - y = 0 \\ -x - y = 0 \end{matrix}\right\}$ → $-x = y$. Thus, $\lambda = 4$ has eigenvectors of $\beta\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

29. $\begin{vmatrix} 2 & 2 & -1 \\ -4 & -7 & -4 \\ 0 & 5 & k-8 \end{vmatrix}$ = 0 → 2 $\begin{vmatrix} -7 & -4 \\ 5 & k-8 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 \\ 5 & k-8 \end{vmatrix} + 0 = 0$ → 2[-7(k-8) +20] +4[2(k-8) + 5] = 0
→ -14k +112+40+8k-64+20 = 0 → 6k = 108 → k= 18

30. $< 3, 5 - 7 > \cdot < -4, 3, 2 > = 3(-4) + 5(3) - 7(2) = -11$