Trigonometry Alpha Test Solutions - MAO National Convention 2014

1. B. By long dividing, $\frac{123456}{360} = 342\frac{336}{360}$. Thus, the angle rotates 342 full times and terminates in Quadrant 4. An angle coterminal with this angle is $336°$, the remainder.

2. D. $\sec\theta > 0$ in Quadrants I and IV. Of these two Quadrants, $\tan\ \theta < 0$ in Quadrant IV.

3. D. Write as $\sin\left(4x - \frac{\pi}{6}\right) = \frac{1}{2}$. Let $u = 4x - \frac{\pi}{6}$. Solving $\sin(u) = \frac{1}{2}$ yields $u = \frac{\pi}{6} + 2\pi k$ or $u = \frac{5\pi}{6} + 2\pi k$ for any integer k. Back substituting, we find $x = \frac{\pi}{12} + \frac{\pi}{2}k$ or $u = \frac{\pi}{4} + \frac{\pi}{2}k$. We see to be in the interval $[0,2\pi)$, $k = 0, 1, 2,$ or 3 resulting in 8 total solutions.

4. A. For simplicity, assume the sides measures 3, 5, 6. By Law of Cosines, $6^2 = 3^2 + 5^2 - 2(3)(5)\cos x$, where x is the measure of the largest angle. Simplifying, $30\cos x = -2$, for which $\cos x = -\frac{1}{15}$.

5. D. The first statement is an identity as seen by multiplying by the conjugate: $\frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} = \frac{\sin x(1-\cos x)}{1-\cos^2 x} = \frac{\sin x(1-\cos x)}{\sin^2 x} = \frac{1-\cos x}{\sin x}$.
   The second statement is true as $\frac{\cot x}{\csc x} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \cos x$.
   The third statement is true since:
   $\sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = (1)(-\cos 2x) = -\cos 2x$
   The fourth statement is false since $1 + \tan^2 x = \sec^2 x$, not the difference.

6. C. $\sin(x - y) = \sin x \cos y - \sin y \cos x = \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{33}{65}$.

7. A. $|\sec x| < \sqrt{2}$ is equivalent to $\cos x > \frac{1}{\sqrt{2}}$ or $\cos x < -\frac{1}{\sqrt{2}}$. For $0 < x < 2\pi$, $\cos x > \frac{1}{\sqrt{2}}$ is satisfied for x in $(0, \frac{\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi)$. Similarly, $\cos x < -\frac{1}{\sqrt{2}}$ for x in $(\frac{3\pi}{4}, \frac{5\pi}{4})$. Thus, we see there is a probability of $\frac{\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{2}}{2\pi} = 50\%$.

8. E. $\sin(\cos^{-1}(\tan(\sin^{-1}\frac{\sqrt{2}}{2}))) = \sin(\cos^{-1}(\tan(\frac{\pi}{4}))) = \sin(\cos^{-1}(1)) = \sin(0) = 0$

9. C. The linear speed of the tip of the hand is $\frac{1\ rev}{60\ mins} \cdot \frac{2\pi(8)cm}{1\ rev} = \frac{16\pi}{60}\frac{cm}{min} = \frac{4\pi}{15}\frac{cm}{min}$.
   Note the linear speed is constant at any time!

10. C. The graph shows a period of 4, an amplitude of 2, and a vertical shift of 1 unit downward. C is correct since the negative leading coefficient would cause a reflection. While choice D has the correct period, amplitude, and vertical shift, the phase shift is the wrong direction (right instead of left).

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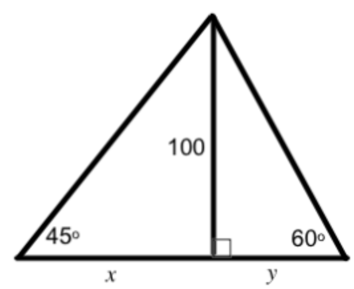
11. C. The area of the triangle can be computed as $\frac{1}{2}(20)(14)\sin A$ where A is the included angle between the given sides. The maximum of $\sin A$ is 1 so the maximum area is $\frac{1}{2}(20)(14)(1) = 140$.

12. C. By De Moivre's Theorem, note $(1-i)^8 = (\sqrt{2}cis\left(-\frac{\pi}{4}\right))^8 = \sqrt{2}^8 cis(-2\pi) = 16cis0 = 16$. Thus, $1 - i$ is one of the eighth roots.

13. C. Apply $\cos 2\theta = 2\cos^2\theta - 1$ to get $2\cos^2\theta + \cos\theta - 1 = 0$. Factoring, we see that $(2\cos\theta - 1)(\cos\theta + 1) = 0$. Thus, $\cos\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ or $\cos\theta = -1$ when $\theta = \pi$. The sum of solutions is $\frac{\pi}{3} + \frac{5\pi}{3} + \pi = 3\pi$.

14. B. This question refers to the Ambiguous Case of the Law of Sines. The height of the triangle would be $h = 16\sin 30° = 8$. For two possible triangles to exist, we require side length a to satisfy $h < a < b$ or $8 < a < 16$. The only option in the choices is $a = 10$.

15. C. Draw a picture such as the one shown. We see that $x = 100\cot 45° = 100$ and $y = 100\cot 60° = \frac{100}{\sqrt{3}}$. Their distance apart is $x + y = 100\left(1 + \frac{1}{\sqrt{3}}\right)$. This approximately equal 100(1.58) or 158 feet.



16. D. First, the domain of $y = \cos^{-1} x$ is [-1,1], thus, $-1 \leq x - \frac{1}{2} \leq 1$ so we see $-\frac{1}{2} \leq x \leq \frac{3}{2}$. For the square root to be defined, we require $\frac{\pi}{2} - \cos^{-1}(x - \frac{1}{2}) \geq 0$, so $\frac{\pi}{2} \geq \cos^{-1}(x - \frac{1}{2})$ so $0 \leq x - \frac{1}{2}$ so $x \geq \frac{1}{2}$. The intersection of these two domain restrictions is the interval $[\frac{1}{2}, \frac{3}{2}]$ so $a + b = \frac{1}{2} + \frac{3}{2} = 2$.

17. B. $\sec\frac{8\pi}{3}\csc\frac{7\pi}{2} + \tan\frac{11\pi}{6}\cos\frac{5\pi}{6} = (-2)(-1) + \left(\frac{-\sqrt{3}}{3}\right)\left(\frac{-\sqrt{3}}{2}\right) = 2 + \frac{1}{2} = \frac{5}{2}$.

18. D. $\tan\left(x - \frac{\pi}{2}\right) \cdot \frac{\sec^2 x - \tan^2 x}{\csc(\pi - x)} = \tan\left(-\left(\frac{\pi}{2} - x\right)\right)\frac{1}{\csc(x)} = -\tan\left(\frac{\pi}{2} - x\right)\sin x = -\cot x \sin x = -\cos x$.

19. A. The amplitude is $\frac{\pi}{8}$ and the period is $\frac{2\pi}{\frac{\pi}{2}} = 4$ so the product is $\frac{\pi}{8} \cdot 4 = \frac{\pi}{2}$.

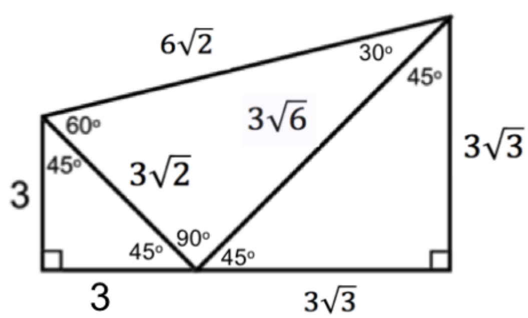
20. A. Use a sum-to-product formula: $\cos(4x) + \cos(2x) = 2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2}$ so our new equation becomes $2\cos 3x \cos x = 0$. For $0 \leq x \leq \pi$, $\cos 3x = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$. Further, for $0 \leq x \leq \frac{\pi}{2}$, $\cos x = 0$ when $x = \frac{\pi}{2}$ only. The sum of solutions is then $\frac{\pi}{6} + \frac{\pi}{2} + \frac{5\pi}{6} = \frac{3\pi}{2}$.

21. C. Using the Law of Sines, $\frac{\sin 2A}{b} = \frac{\sin A}{a}$. Using a double angle identity,

$\frac{2\sin A\cos A}{b} = \frac{\sin A}{a}$. Since $\sin A \neq 0$, we cancel this common factor to see that $\cos A = \frac{b}{2a}$. Since $\frac{b}{a} = \frac{3}{2}$, then $\cos A = \frac{b}{2a} = \frac{3}{4}$.

22. A. We see the radius of the circle must be $r = \frac{18\pi}{2\pi} = 9$.
By special right triangles, we see the height of the triangle must be $h = \frac{9}{2}$ and the central angle of the sector is $120°$. Thus, the area of the triangle is $\frac{1}{2}\left(9\sqrt{3}\right)\frac{9}{2} = \frac{81\sqrt{3}}{4}$ and the area of the sector is $\frac{120}{360}(\pi 9^2) = 27\pi$. The desired area is the difference of these areas, namely, $27\pi - \frac{81\sqrt{3}}{4} = \frac{27}{4}(4\pi - 3\sqrt{3})$ square units.

23. C. Use special right triangles to find the following lengths:
We deduce that $x = 6\sqrt{2},\ y = 3\sqrt{3}$, thus $\frac{x}{y} = \frac{6\sqrt{2}}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{6}}{9} = \frac{2\sqrt{6}}{3}$.

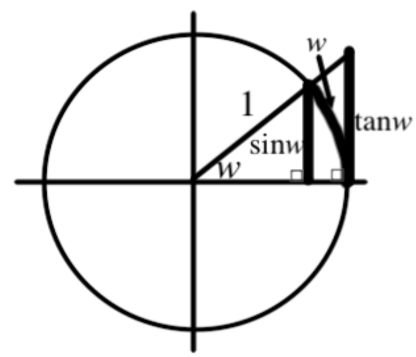


24. C. The maximum and minimum values of $y = \sin x$ are 1 and -1, respectively. Thus, when $|x| > 100\pi$, there will not be any intersections between the graphs. Next, note $\frac{100\pi}{2\pi} = 50$, thus, exactly 50 complete cycles of $y = \sin x$ will occur on either side of the *y*-axis before the linear function is beyond the range of sine. Each cycle (besides the first which only contributes 1 intersection) will contribute 2 intersections, thus there are 49(2) + 1 = 99 intersections both to the right and left of the *y*-axis. Lastly, there is one intersection when *x* = 0 accounting for 2(99) + 1 = 199 intersections.

25. B. The radii measure 6, 7, and 8 and the lengths of the triangle are then 13, 14, and 15. Using Heron's Formula, the semiperimeter is $s = \frac{13+14+15}{2} = 21$ and the triangle's area is $\sqrt{21(21-13)(21-14)(21-15)} = \sqrt{(3)(7)(4)(2)(7)(3)(2)}$ when factored. This is simply $\sqrt{(9)(49)(4)(4)} = (3)(7)(2)(2) = 84$.

26. B. Apply the Binomial Expansion Theorem to write the expression as $(\sin^2 x + \cos^2 x)^3 = 1^3 = 1$. Since this expression evaluates to a constant, the maximum value is 1.

27. B. Use a co-function identity to write $\cos\left(\frac{\pi}{2} - 3x\right) = \cos 7x$. If $\cos u = \cos v$ then either $u = 2\pi k + v$ for some integer k, or $u = 2\pi k - v$ for some integer k. Hence, $\frac{\pi}{2} - 3x = 2\pi k + 7x$ for which $x = \frac{\frac{\pi}{2} - 2\pi k}{10}$. When k = 0, -1 or -2, $x = \frac{\pi}{20}, \frac{\pi}{4}, \frac{9\pi}{20}$ respectively. Also, $\frac{\pi}{2} - 3x = 2\pi k - 7x$ for which $x = \frac{2\pi k - \frac{\pi}{2}}{4}$. When k = 1 or 2, $x = \frac{3\pi}{8}, \frac{7\pi}{8}$ respectively. Thus, the first four solution in increasing order are $\frac{\pi}{20}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{9\pi}{20}$ and the third element is $\frac{3\pi}{8}$.



28. A. Consider the Unit Circle (radius 1) and the acute angle w drawn in standard position. As shown in the diagram, the bolded lengths represent sin$w$, w, and tan$w$, where $\tan w > w > \sin w$, so $y > z > x$.

29. A. Note that $1 = \tan(22° + 23°) = \frac{\tan 22° + \tan 23°}{1 - \tan 22° \tan 23°}$. Then $1 = \frac{\tan 22° + \tan 23°}{1 - \tan 22° \tan 23°}$, thus $1 - \tan 22° \tan 23° = \tan 22° + \tan 23°$. Thus, we see $1 = \tan 22° + \tan 23° + \tan 22° \tan 23°$.

30. C. Use symmetry of the Unit Circle. Since $\cos x = -\cos(\pi - x)$, then the cosine values of $\frac{\pi}{1000}, \frac{2\pi}{1000}, \ldots \frac{499\pi}{1000}$ will cancel with those cosine values of $\frac{999\pi}{1000}, \frac{998\pi}{1000}, \frac{501\pi}{1000}$, respectively . In a similar fashion, those cosine values between angles in Quadrants III and IV will offset. Of the remaining angles, $\cos\frac{500\pi}{1000} = 0$, $\cos\frac{1500\pi}{1000} = 0$, and $\cos\frac{1000\pi}{1000} = -1$, so the entire sum evaluates to -1.