# BC Calculus Alpha Test Solutions - MAO National Convention 2014

1. C. The points (3, 5) and (-1, -3) are on the tangent line, thus the tangent slope at $x = -1$ is $\frac{5-(-3)}{3-(-1)} = 2$. Thus g'(-1) = 2.

2. C. $\lim_{x\to-\infty} \frac{7+4x-6x^2}{\sqrt{x^4\left(4+\frac{2}{x^3}-\frac{1}{x^4}\right)}} = \lim_{x\to-\infty} \frac{x^2\left(-6+\frac{4}{x}+\frac{7}{x^2}\right)}{x^2\sqrt{\left(4+\frac{2}{x^3}-\frac{1}{x^4}\right)}} = \lim_{x\to-\infty} \frac{\left(-6+\frac{4}{x}+\frac{7}{x^2}\right)}{\sqrt{\left(4+\frac{2}{x^3}-\frac{1}{x^4}\right)}} = \frac{-6}{\sqrt{4}} = -3$

3. C. $x'(t) = 2e^{2t} + 2\cos t$, thus $x'(0) = 4$ . Also $y'(t) = \sec^2 t$ so $y'(0) = 1$. Thus, $m = \left.\frac{dy}{dx}\right|_{t=0} = \frac{y'(0)}{x'(0)} = \frac{1}{4}$. Also note that when $t = 0$, $x = 1$ and $y = 3$. Thus, the equation of the tangent line is $y - 3 = \frac{1}{4}(x-1)$ and when written in standard form, $x - 4y = -11$ .

4. A. Note $\ln(xy) = \ln(x) + \ln(y)$. Now differentiate both sides with respect to $x$, noting differentiation rules when needed: $12x^2 - 2y^2 - 4xy\frac{dy}{dx} + \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$. Multiply both sides of the equation by $xy$ to yield $12x^3y - 2xy^3 - 4x^2y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$. Now solve for $\frac{dy}{dx}$ to yield $\frac{dy}{dx} = \frac{12x^3y-2xy^3+y}{4x^2y^2-x}$.

5. C. Note $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ which converges for all real $x$. Letting $x = -\pi\ln 2$ results in the given series, so the value is $e^{-\pi\ln 2} = e^{\ln 2^{-\pi}} = \frac{1}{2^\pi}$.

6. D. I: Complete the square to write $\int \frac{3}{9x^2-6x+2}dx = \int\frac{3}{(3x-1)^2+1} = \arctan(3x-1) + C$. Thus, $\int_1^\infty \frac{3}{9x^2-6x+2}dx = \lim_{t\to\infty}\arctan(3x-1) - \arctan(2) = \frac{\pi}{4} - \arctan(2)$. Converges!

II: Use partial fractions to show $\int \frac{x+7}{x^2-x-6}dx = \int\frac{2}{x-3} - \frac{1}{x+2}dx = 2\ln|x-3| - \ln|x+2| + C$ . Using log properties, this equals $\ln\left|\frac{(x-3)^2}{x+2}\right| + C$. As $x\to\infty$ this Diverges!

III: Let $u = x - 1$ to get $\int_0^1 2u^{-\frac{1}{3}}du = \left.3u^{\frac{2}{3}}\right|_{u=0}^{u=1} = 3$. Converges!

7. B. Use the Fundamental Theorem and Chain Rule to note: $f'(x) = \frac{\tan^{-1}\sqrt{3x}}{\sqrt{1+x}}\frac{1}{2\sqrt{x}} - \frac{\tan^{-1}(\sqrt{3}\ln x)}{\sqrt{1+(\ln x)^2}}$ . Thus, $f'(1) = \frac{\tan^{-1}\sqrt{3}}{\sqrt{2}}\frac{1}{2} - \frac{\tan^{-1}(\sqrt{3}\ln 1)}{\sqrt{1+(\ln 1)^2}} = \frac{\frac{\pi}{3}}{2\sqrt{2}} - 0 = \frac{\pi\sqrt{2}}{12}$ .

8. D. Use integration parts to show $\int xe^{-2x}\,\mathrm{dx} = \frac{-1}{4}e^{-2x}(2x+1)$. Using the Fundamental Theorem, $\int_0^{\ln 2} xe^{-2x}\,dx = \frac{-1}{4}e^{-2\ln 2}(2\ln 2+1) + \frac{1}{4}e^0(0+1) = \frac{-1}{16}(\ln 4 + 1) + \frac{1}{4} = \frac{3-\ln 4}{16}$. The average value is then $\frac{3-\ln 4}{16\ln 2}$ .

9. C. $h'(x) = \frac{f'(x^3)(3x^2)g(2x) - g'(2x)(2)f(x^3)}{(g(2x))^2}$ so $h'(-2) = \frac{f'(-8)(12)g(-4)-g'(-4)(2)f(-8)}{(g(-4))^2}$. Use the table to find $h'(-2) = \frac{(-1)(12)(2)-(-2)(2)(3)}{(2)^2} = \frac{-24+12}{4} = -3$

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10. B. Use L'Hopital's Rule:
$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} = \lim_{x\to 0}\frac{2\sec^2 x\tan^2 x + 2\sec^3 x}{6} = \frac{1}{3}$.

11. A. Note $y' = \frac{\sec x\tan x}{\sec x} = \tan x$. $L = \int_0^{\frac{\pi}{4}}\sqrt{1+(\tan x)^2}dx = \int_0^{\frac{\pi}{4}}\sec x\, dx = \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}}$ $= \ln(\sqrt{2}+1)$.

12. C. The Lagrange Error Bound says $|E(x)| \le \frac{t^5}{5}$, namely, the next term, for $0 \le t \le x$. When $x = \frac{1}{2}$, $\left|E\left(\frac{1}{2}\right)\right| \le \frac{\frac{1}{32}}{5} = \frac{1}{160}$.

13. A. $W = \int_0^{\pi}\sin x + 4x\, dx = -\cos x + 2x^2|_0^{\pi} = 2 + 2\pi^2$.

14. B. Use the Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)}{3^{n+1}(n+1)^2}|2x+1|^{n+1}\cdot\frac{3^n(n)^2}{n+1}\cdot\frac{1}{|2x+1|^n} < 1$
Thus, $\lim_{n\to\infty}\frac{n^2(n+2)}{3(n+1)\text{^}3}|2x+1| < 1$ so $|2x+1| < 3$, or $-2 < x < 1$. However, we must check the endpoints. When $x = -2$, the series is $\sum_{n=1}^{\infty}\frac{(n+1)}{n^2}(-1)^n$, which converges by the Alternating Series Test. When $x = 1$, the series is $\sum_{n=1}^{\infty}\frac{(n+1)}{n^2}$, which diverges by comparison to the Harmonic Series. Thus, our convergence interval is exactly $-2 \le x < 1$.

15. C. Solve $0 = 2 + 4\cos\theta$ to get $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. The area element $dA = \frac{1}{2}r^2 d\theta$, so the area is $\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}(2+4\cos\theta)^2\, d\theta$.

16. B. $\sum_{n=1}^{\infty}\frac{\sin n}{n^4}$ converges absolutely by comparison to *p*-series. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by Alternating Series test, but does not absolutely since in absolute value it is the Harmonic Series. $\sum_{n=1}^{\infty}\cos(n+3)$ diverges by the *n*th term test. $\sum_{n=1}^{\infty}\left(\frac{5n+2n^3}{8n^3-1}\right)^n$ converges by the root test.

17. D. Separate variables: $\frac{dy}{y} = \frac{dx}{\sqrt{1-x^2}}$. Integrating both sides, $\ln y = \arcsin(x) + C$. Using the initial condition, we find $C = -\frac{\pi}{2}$ and thus $y = e^{\arcsin(x)-\frac{\pi}{2}}$. Hence, $y(0) = e^{\arcsin(0)-\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$.

18. B. Look at maxima/minima, increasing/decreasing, and concavity.

19. C. Differentiability implies continuity so A and B must be true. D is just a restatement of being differentiable (from the right).

20. B. $\ln y = -x\ln(\sin 2x)$, thus $\frac{y'}{y} = -\ln(\sin 2x) - \frac{2x\cos 2x}{\sin 2x}$. Multiply over by $y$ to get the answer.

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21. B. The natural logarithm follows the behavior of the slope fields.

22. A. Volume = $2\pi\int_0^1 x\sqrt{x}dx = \frac{4\pi}{5}x^{\frac{5}{2}}\Big|_0^1 = \frac{4\pi}{5}$.

23. D. $f'(x) = 2x^4 - \frac{4}{3}x^3 + 2$. $f''^{(x)} = 8x^3 - 4x^2 = 4x^2(2x-1) = 0$ when $x = 0, \frac{1}{2}$. However, note that the root of 0 has a multiplicity of 2 so inflection will not change here. There is only one inflection point at $x = \frac{1}{2}$.

24. B. $V = \frac{4}{3}\pi r^3$ so $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Since the radius is $\frac{2}{2\pi} = \frac{1}{\pi}$, we have $-4 = 4\pi\frac{1}{\pi^2}\frac{dr}{dt}$. Solving, $\frac{dr}{dt} = -\pi$ cm per minute.

25. D. Note for $x \neq 3, f(x) = \frac{(x+3)(x-3)}{x-3} = x+3$. $\lim_{x\to 3} f(x) = 3+3 = 6 = f(3)$ so $f$ is continuous at $x = 3$. Thus, really, $f(x) = x+3$ which is differentiable everywhere.

26. A. Using $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, we have $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$. Lastly, multiply through by $x$ to get $\frac{x}{1+x^2} = x - x^3 + x^5 - x^7 + \cdots$.

27. B. For the velocity to be increasing, we require $x''(t) > 0$, namely, when $x(t)$ is concave up. This occurs on the interval [0, 4].

28. B. The area of a rectangle is base times height. Thus, we multiply the base and height for each of the three rectangles, where the height is the function evaluated at the right endpoint of each interval: (3-2)(12) + (7-3)(14) + (10-7)(4) = 12 + 56 + 12 = 80.

29. A. $\int \sin^3 x \cdot \cos^2 x\, dx = \int \sin x \cdot \sin^2 x \cdot \cos^2 x\, dx = \int \sin x \cdot (1 - \cos^2 x) \cdot \cos^2 x\, dx$.
$= \int \sin x \cdot (\cos^2 x - \cos^4 x)dx$. Let $u = \cos x$ and note $dx = \frac{-du}{\sin x}$. This yields
$-\int u^2 - u^4 du = -\frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$.

30. B. Since $f$ is continuous and differentiable on [2,12], the Mean Value Theorem applies. Note that we are guaranteed some $c$ in [2,12] such that $f'(c) = \frac{f(12)-f(2)}{10}$. Solving for $f(12)$: $f(12) = 10f'(c) + f(2)$. Substituting what we are given, $f(12) \leq 10(5) + 8 = 58$.