Interesting Mathematical Problems to Ponder

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Exercise 1. Factor

\[ a^3 + b^3 + c^3 - 3abc. \]

Solution: Consider the monic third degree polynomial whose zeros are \( a, b, c \):

\[ x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc. \]

Then

\[
\begin{align*}
a^3 - (a + b + c)a^2 + (ab + bc + ca)a - abc &= 0 \\
b^3 - (a + b + c)b^2 + (ab + bc + ca)b - abc &= 0 \\
c^3 - (a + b + c)c^2 + (ab + bc + ca)c - abc &= 0.
\end{align*}
\]

Adding up these three equalities yields

\[
a^3 + b^3 + c^3 - (a + b + c)(a^2 + b^2 + c^2) + (ab + bc + ca)(a + b + c) - 3abc = 0.
\]

Hence

\[
a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca). \tag{1}
\]

Another way to obtain the identity (1) is to consider the determinant

\[
D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}
\]

Expanding \( D \) we have

\[
D = a^3 + b^3 + c^3 - 3abc.
\]

On the other hand, adding up all columns yields

\[
D = \begin{vmatrix} a + b + c & b & c \\ a + b + c & a & b \\ a + b + c & c & a \end{vmatrix} = (a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}
\]

\[
= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).
\]
Note that the expression
\[ a^2 + b^2 + c^2 - ab - bc - ca \]
can be also written as
\[ \frac{1}{2} [(a - b)^2 + (b - c)^2 + (c - a)^2]. \]

We obtain another version of the identity (1):
\[ a^3 + b^3 + c^3 - 3abc = \frac{1}{2} (a + b + c) [(a - b)^2 + (b - c)^2 + (c - a)^2]. \quad (2) \]

This form leads to a short proof of the AM-GM inequality for three variables. Indeed, from (2) it is clear that if \( a, b, c \) are nonnegative, then \( a^3 + b^3 + c^3 \geq 3abc \). Now, if \( x, y, z \) are positive numbers, taking \( a = \sqrt[3]{x}, b = \sqrt[3]{y}, c = \sqrt[3]{z} \) yields
\[ \frac{x + y + z}{3} \geq \sqrt[3]{xyz}, \]
with equality if and only if \( x = y = z \).

**Exercise 2.** Find the minimum of \( 3^{x+y}(3^{x-1} + 3^{y-1} - 1) \) over all pairs \((x, y)\) of real numbers.

**Solution:** Let \( f(x, y) = 3^{x+y}(3^{x-1} + 3^{y-1} - 1) \). We have
\[ 3f(x, y) + 1 = 3^{2x+y} + 3^{x+2y} + 1 - 3 \cdot 3^{x+y}, \]
which is of the form \( a^3 + b^3 + c^3 - 3abc \), where \( a = \sqrt[3]{3^{2x+y}}, b = \sqrt[3]{3^{x+2y}}, \) and \( c = 1 \) are all positive real numbers. From (2) it follows that \( 3f(x, y) + 1 \geq 0 \) for all \( x, y \in \mathbb{R} \), with equality if and only if \( x = y = 0 \). Hence the minimum of \( f(x, y) \) is \(-\frac{1}{3}\).

The same conclusion follows directly from the AM-GM inequality, because
\[ 3^{2x+y-1} + 3^{x+2y-1} + 3^{-1} \geq 3\sqrt[3]{3^{2x+y-1} + 3^{x+2y-1}} - 3^{x+y} = 3^{x+y}, \]

implying
\[ 3^{2x+y-1} + 3^{x+2y-1} - 3^{x+y} \geq -\frac{1}{3}. \]

Hence
\[ 3^{x+y}(3^{x-1} + 3^{y-1} - 1) \geq -\frac{1}{3}, \]

for all real numbers \( x, y \), with equality if and only if \( 2x + y - 1 = x + 2y - 1 = -1 \), i.e. \( x = y = 0 \).
Exercise 3. If \( a + b + c = 0 \), then \( a^3 + b^3 + c^3 = 3abc \).

Solution: Follows immediately from (2).

**Problem 1.** Simplify

\[
(x + 2y - 3z)^3 + (y + 2z - 3x)^3 + (z + 2x - 3y)^3.
\]

Solution: Setting \( x + 2y - 3z = a \), \( y + 2z - 3x = b \), \( z + 2x - 3y = c \), we have \( a + b + c = 0 \), and from Exercise 2 it follows that \( a^3 + b^3 + c^3 = 3abc \).

Hence the given expression is equal to

\[
3(x + 2y - 3z)(y + 2z - 3x)(z + 2x - 3y).
\]

**Problem 2.** Let \( a, b, c \) be complex numbers. Prove that \( a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2 \) if and only if \( a = b, orb = c,orc = a \).

Solution: Because \((a - b) + (b - c) + (c - a) = 0\),

\[
(a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a),
\]

so assuming \( a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2 \) yields

\[
3(a - b)(b - c)(c - a) = a^3 - b^3 + b^3 - c^3 + c^3 - a^3 - 3[(a^2b + b^2c + c^2a) - (ab^2 + bc^2 + ca^2)] = 0.
\]

Then \( a = b \), or \( b = c \), or \( c = a \). The converse follows immediately.

The conclusion of the problem follows directly from

\[
(a - b)(b - c)(c - a) = abc - (a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) - abc = (ab^2 + bc^2 + ca^2) - (a^2b + b^2c + c^2a).
\]

**Problem 3.** Let \( x, y, z \) be distinct real numbers. Prove that

\[
\sqrt[3]{x - y} + \sqrt[3]{y - z} + \sqrt[3]{z - x} \neq 0.
\]

Solution: Assume the contrary, and let \( \sqrt[3]{x - y} = a \), \( \sqrt[3]{y - z} = b \), \( \sqrt[3]{z - x} = c \). Then \( a + b + c = 0 \), and, from Exercise 2, \( a^3 + b^3 + c^3 = 3abc \). This yields

\[
0 = (x - y) + (y - z) + (z - x) = 3\sqrt[3]{(x - y)(y - z)(z - x)} \neq 0,
\]
a contradiction. The problem is solved.

**Problem 4.** Let \( r \) be a real number such that \( \sqrt[3]{r} - \frac{1}{\sqrt[3]{r}} = 2 \). Find \( r^3 - \frac{1}{r^3} \).

(UWW Mathmeet, 2003)

Solution: With \( a = \sqrt[3]{r} \), \( b = -\frac{1}{\sqrt[3]{r}} \), \( c = -2 \), we have again \( a + b + c = 0 \), hence \( a^3 + b^3 + c^3 = 3abc \). This yields

\[
r - \frac{1}{r} - 8 = 3\sqrt[3]{r} \left(-\frac{1}{\sqrt[3]{r}} \right) (-2),
\]
or, equivalently,

\[ r - \frac{1}{r} - 14 = 0. \]

By applying the result in Exercise 2 again, we get

\[ r^3 - \frac{1}{r^3} - 2744 = 3r \left( -\frac{1}{r} \right)(-14), \]

or

\[ r^3 - \frac{1}{r^3} = 2744 + 42 = 2786. \]

**Problem 5.** Show that if the numbers \( abc, bca, cab \) are divisible by \( n \), then so is \( a^3 + b^3 + c^3 - 3abc \).

**Solution:** We have seen that

\[ a^3 + b^3 + c^3 - 3abc = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \frac{1}{100} \begin{vmatrix} 100b + 10c + a & b & c \\ 100a + 10b + c & a & b \\ 100c + 10a + b & c & a \end{vmatrix} = \begin{vmatrix} bca & b & c \\ abc & a & b \\ cab & c & a \end{vmatrix}, \]

and the conclusion follows.

**Problem 6.** The number of ordered pairs of integers \((m, n)\) such that \( mn \geq 0 \) and \( m^3 + 99mn + n^3 = 33^3 \) is

a) 2   b) 3   c) 33   d) 35   e) 99.

*(AHSME 1999)*

**Solution:** Write the given relation as

\[ m^3 + n^3 + (-33)^3 - 3mn(-33) = 0. \]

From the identity (2) it follows that

\[ (m + n - 33) \left[ (m - n)^2 + (m + 33)^2 + (n + 33)^2 \right] = 0. \]

The equation \( m + n = 33 \), along with the condition \( mn \geq 0 \), yields 34 solutions: \((k, 33 - k), k = 0, 1, \ldots, 33\). The second factor is equal to zero only when \( m = n = -33 \), giving the 35th solution.