Solutions

1. The key insight is to note that $2^n \times 5^n = 10^n$. From here, it is clear that if 2^n and 5^n begin with the same digit, that digit must be either 1 (if 2^n and 5^n are both powers of 10), or 3. Since n > 0, we reject the former case; hence the first digit must be 3. (If the first digit of 2^n is less than 3, then the first digit of 5^n must be greater than or equal to 3. If the first digit of 2^n is greater than 3, then the first digit of 5^n must be less than 3.) More formally, if 2^n and 5^n begin with the digit d, then:

- $d \cdot 10^r < 2^n < (d + 1) \cdot 10^r$, and
- $d \cdot 10^s < 5^n < (d + 1) \cdot 10^s$, for some non-negative integers r and s.

(We have strict inequality because, for n > 0, $2^n \equiv 0 \pmod{10} \Rightarrow 2^n \equiv 0 \pmod{5}$, which is impossible by the Fundamental Theorem of Arithmetic. Similarly for $5^n \equiv 0 \pmod{10}$.)
Multiplying the inequalities, we obtain $d^2 \cdot 10^{r+s} < 10^n < (d + 1)^2 \cdot 10^{r+s}$.
Hence $1 \leq d^2 < 10^{n-r-s} < (d + 1)^2 \leq 100$. (Since d is a decimal digit.)
It follows that $n - r - s = 1$, so that $d^2 < 10 < (d + 1)^2$, and d = 3.
Therefore, if the numbers 2^n and 5^n (where n is a positive integer) start with the same digit, then that digit must be **3**.

2. The probability, p(n), of getting a free ticket when you are the *n*th person is line is:

(probability that none of the first n−1 people share a birthday) · (probability that you share a birthday with one of the first n−1 people)

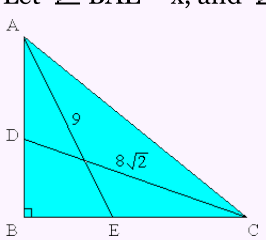
So p(n) = [1 · 364/365 · 363/365 · …· (365−(n−2))/365] · [(n−1)/365]

We seek the least n such that p(n) > p(n+1), or p(n)/p(n+1) > 1 (since p(n) > 0.)
This will locate the first (and only) maximum of the probability distribution function; i.e., its mode.
p(n)/p(n+1) = 365/(366−n) · (n−1)/n

Now, p(n)/p(n+1) > 1 implies $365n - 365 > 366n - n^2$, and so $n^2 - n - 365 > 0$.
An approximate factorization of this quadratic equation gives: (n + 18.6)(n − 19.6) > 0.
Rejecting the negative region, inequality is satisfied if n > 19.6
Therefore the position in line that gives you the best chance of being the first duplicate birthday is **20th**.

3. Following is one method of solution; others exist.

Let ∠BAE = x, and ∠BCD = y.



AB = 9 cos x = AC cos 2x.   BC = $8\sqrt{2}$ cos y = AC cos 2y.
Eliminating AC:
(9 cos x) / (cos 2x) = ($8\sqrt{2}$ cos y) / (cos 2y). y = 45° − x. Also, 2y = 90° − 2x, and so cos 2y = sin 2x.
Therefore: (9 cos x) / (cos 2x) = ($8\sqrt{2}$ cos (45°−x)) / (sin 2x).

Using trigonometric identity cos(a − b) = cos a · cos b + sin a · sin b:
cos (45°−x) = (cos x + sin x) / $\sqrt{2}$. Rearranging: tan 2x = 8(cos x + sin x) / 9 cos x.

Using trigonometric identity $\tan 2a = 2 \tan a / (1 - \tan^2 a)$, and letting $t = \tan x$:
$2t / (1 - t^2) = 8(1 + t)/9$. Therefore $9t/4 = (1 + t)(1 - t^2) = 1 + t - t^2 - t^3$. Hence $t^3 + t^2 + (5/4)t - 1 = 0$.

By inspection, one root is $t = 1/2$. Therefore $(t - 1/2)(t^2 + 3t/2 + 2) = 0$.

The quadratic factor has no real roots (since $(3/2)^2 - 4{\cdot}1{\cdot}2 < 0$), and so $t = 1/2$ is the only real root.
$AC = 9 \cdot \cos x / \cos 2x$. Using trigonometric identities $\cos x = 1 / \sqrt{(1 + t^2)}$, $\cos 2x = (1 - t^2)/(1 + t^2)$:
$AC = 9 \cdot \sqrt{(1 + t^2)} / (1 - t^2)$.

Therefore $AC = 9 \cdot (\sqrt{5}/2) / (3/4) =$ **$6\sqrt{5}$ inches.**

4. The terms of a sequence of positive integers satisfy $a_{n+3} = a_{n+2}(a_{n+1} + a_n)$, for $n = 1, 2, 3, \ldots$.
If $a_6 = 8820$, what is a_7?
Letting $a_1 = x$, $a_2 = y$, and $a_3 = z$, from the recurrence relation we obtain

$a_4 = z(y + x)$,
$a_5 = z(y + x)(z + y)$,
$a_6 = z(y + x)(z + y)[z(y + x) + z] = z^2(y + x)(y + x + 1)(z + y)$.

Hence we have $z^2(y + x)(y + x + 1)(z + y) = 8820 = 2^2 \times 3^2 \times 5 \times 7^2$. A certain amount of trial and error is now required. The goal is to minimize the need for trial and error by applying mathematical constraints.
By the Fundamental Theorem of Arithmetic, the factors on the left-hand side are the same as those on the right-hand side of the equation. In particular, two of the factors on the left-hand side are consecutive integers, and therefore must be relatively prime. (Any divisor of n and n + 1 must divide n + 1 – n = 1.) This enables us to determine candidate values for (y + x) and (y + x + 1) -- by partitioning the prime factors 2, 3, 5, 7 -- more easily than if we had to obtain a complete list of the 54 factors of 8820.
Additionally, we note that if $z = 1$, then $(y + 1)(y + x)(y + x + 1) = 8820$, with $y + 1 \leq y + x < y + x + 1$, so that we must have $y + x + 1 > 8820^{1/3}$; that is, $y + x \geq 20$. This enables us to exclude the case $z = 1$ from many of the candidate factorizations below.

Candidate factorizations of 8820

(y + x)	**(y + x + 1)**	**$z^2(z + y)$**	**Deductions**
2	3	$2 \times 3 \times 5 \times 7^2$	$y + x = 2 \Rightarrow y = 1$. $1 < z^2 \mid 2 \times 3 \times 5 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) = 7^2 \times 8$. Contradiction.
3	2^2	$3 \times 5 \times 7^2$	$y + x = 3 \Rightarrow y \leq 2$. $1 < z^2 \mid 2 \times 3 \times 5 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) \leq 7^2 \times 8$. Contradiction.
2^2	5	$3^2 \times 7^2$	$y + x = 4 \Rightarrow y \leq 3$. $1 < z^2 \mid 3^2 \times 7^2 \Rightarrow z = 3, 7$, or 21. Then: • $z = 3 \Rightarrow z^2(z + y) \leq 3^2 \times 6$. Contradiction. • $z = 7 \Rightarrow y = 2, x = 2$. **Solution!** • $z = 21 \Rightarrow z + y = 21$, and so $y = 0$. Contradiction.
5	2×3	$2 \times 3 \times 7^2$	$1 < z^2 \mid 2 \times 3 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) > 7^3$. Contradiction.

Candidate factorizations of 8820			
(y + x)	**(y + x + 1)**	**$z^2(z + y)$**	**Deductions**
2×3	7	$2 \times 3 \times 5 \times 7$	$1 < z^2 \mid 2 \times 3 \times 5 \times 7$ is impossible.
3^2	2×5	2×7^2	$1 < z^2 \mid 2 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) > 7^3$. Contradiction.
2×7	3×5	$2 \times 3 \times 7$	$1 < z^2 \mid 2 \times 3 \times 7$ is impossible.
$2^2 \times 5$	3×7	3×7	$1 < z^2 \mid 3 \times 7$ is impossible.
5×7	$2^2 \times 3^2$	7	$z^2 \mid 7 \Rightarrow z = 1$, and y = 6, x = 29. **Solution!**

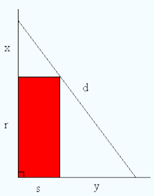
Hence we have two solutions, (x, y, z) = (2, 2, 7) or (29, 6, 1). The two sequences are thus, respectively:

2, 2, 7, 28, 252, 8820, 2469600... , and

29, 6, 1, 35, 245, 8820, 2469600... .

Therefore, if $a_6 = 8820$, a_7 = **2469600.**

5. First of all, we generalize the dimensions of the box to r × s; see below. Let the distance above the box at which the ladder touches the wall be x, and let the corresponding horizontal distance be y. Let the ladder have length d.



Applying Pythagoras' Theorem, $(r + x)^2 + (s + y)^2 = d^2$. By similar triangles, $x/s = r/y$, and so $y = rs/x$.
Hence $(r + x)^2 + (s + rs/x)^2 = d^2$.

Expanding, and collecting terms, we obtain an equation in x:
$f(x) = x^2 + 2rx + (r^2 + s^2 - d^2) + 2rs^2/x + r^2s^2/x^2 = 0$. (1)

Multiplying by x^2, we obtain (since x = 0 is not a root) an equivalent polynomial equation in x:
$p(x) = x^4 + 2rx^3 + (r^2 + s^2 - d^2)x^2 + 2rs^2x + r^2s^2 = 0$. (2)

Clearly $d^2 > r^2 + s^2$, and so by Descartes' Sign Rule, this equation has 2 or 0 positive real roots (counting multiplicity.)
We seek the value(s) of d for which the equation has two identical roots.
p has repeated root a if, and only if, $p(x) = (x - a)^2(x^2 + bx + r^2s^2/a^2)$, for some a and b, to be found. (The constant term in the second quadratic factor is determined by that in the first, given that their product must equal r^2s^2.)

Expanding, $p(x) = x^4 + (b - 2a)x^3 + (a^2 - 2ab + r^2s^2/a^2)x^2 + (a^2b - 2r^2s^2/a)x + r^2s^2 = 0$.
Equating coefficients of x^3 and x with (2), we obtain:

$b - 2a = 2r$,

$a^2b - 2r^2s^2/a = 2rs^2$. (3)

Substituting $b = 2(a + r)$ into (3), multiplying by a, and rearranging, we get
$2a^3(a + r) = 2(a + r)rs^2$.
Since $a + r \neq 0$, we deduce that $a^3 = rs^2$.
Hence $a = r^{1/3}s^{2/3}$, and $b = 2(a + r) = 2r^{1/3}s^{2/3} + 2r$.
Equating coefficients of x^2, we obtain:

$r^2 + s^2 - d^2 = a^2 - 2ab + r^2s^2/a^2$.

$= r^{2/3}s^{4/3} - 4r^{2/3}s^{4/3} - 4r^{4/3}s^{2/3} + r^{4/3}s^{2/3}$.

$= -3r^{2/3}s^{4/3} - 3r^{4/3}s^{2/3}$.

Hence $d^2 = r^2 + s^2 + 3r^{2/3}s^{4/3} + 3r^{4/3}s^{2/3} = (r^{2/3} + s^{2/3})^3$.
(Note the pleasingly symmetrical form: $d^{2/3} = r^{2/3} + s^{2/3}$.)
Finally, substituting $r = 64 = 4^3$ and $s = 27 = 3^3$, we get $d^2 = (4^2 + 3^2)^3 = (5^2)^3$, so that $d = 5^3 = 125$.
Therefore, the length of the ladder so that there is only one position in which it can touch the ground, the box, and the wall, is **125 units.**

6. Let the numbers be a, b, c, d, and e. Then we have

$a + b + c + d + e = 7$
$a^2 + b^2 + c^2 + d^2 + e^2 = 10$

Without loss of generality, we seek the minimum and maximum possible values of e. Rewrite the above as

$a + b + c + d = 7 - e$
$a^2 + b^2 + c^2 + d^2 = 10 - e^2$

We will use the Cauchy-Schwarz inequality to derive an inequality involving only e.
Letting $\mathbf{x} = (1, 1, 1, 1)$, $\mathbf{y} = (a, b, c, d)$, by Cauchy-Schwarz we have $|\mathbf{x.y}| \leq (\mathbf{x.x})^{1/2} (\mathbf{y.y})^{1/2}$; thus

$|a + b + c + d| \leq (1^2 + 1^2 + 1^2 + 1^2)^{1/2} (a^2 + b^2 + c^2 + d^2)^{1/2}$, with equality if, and only if, $a = b = c = d$; and hence
$|7 - e| \leq 2(10 - e^2)^{1/2}$

Squaring both sides of this inequality, we obtain

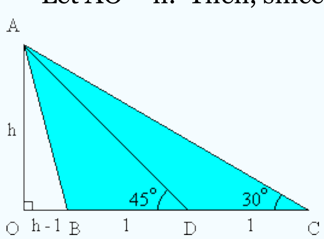
$e^2 - 14e + 49 \leq 4(10 - e^2)$.

Therefore $5e^2 - 14e + 9 = (e - 1)(5e - 9) \leq 0$.
One factor must be positive and the other negative; hence $1 \leq e \leq 1.8$.

We must verify that the minimum and maximum values of e may be attained. From the derivation above, we know that equality occurs if, and only if, $a = b = c = d$. This guides us to:
The minimum of e = 1 is attained when a = b = c = d = 1.5, e = **1.**
The maximum of e = 1.8 is attained when a = b = c = d = 1.3, e = **1.8**.

7. Extend CB to O, so that AO ⊥ CO. (See remark 2, below.) Without loss of generality, let BD = DC = 1.

Let AO = h. Then, since $1 = \cot 45° = OD/h$, we obtain $OB = h - 1$.



$\sqrt{3} = \cot 30° = (h + 1)/h = 1 + (1/h)$. Hence $1/h = \sqrt{3} - 1$. From here, we could note that
$\cot OBA = (h - 1)/h = 1 - (1/h) = 2 - \sqrt{3}$.

This essentially solves the problem, as $\angle OBA = \cot^{-1}(2 - \sqrt{3})$, and then $\angle ABC = 180° - \angle OBA$, and we have expressed $\angle ABC$ in terms of known quantities. However, if we hope to find an exact value in degrees for $\angle ABC$, it is not obvious that $\cot^{-1}(2 - \sqrt{3}) = 75°$. Rather than work backwards, by calculating cot 75° or tan 75°, we pursue an alternative approach below.

Since $1/h = \sqrt{3} - 1$, rationalizing the denominator, we obtain $h = ½(1 + \sqrt{3})$.

Then $AB^2 = h^2 + (h - 1)^2$, by Pythagoras' Theorem

$= 2h(h - 1) + 1$

$= ½(1 + \sqrt{3})(-1 + \sqrt{3}) + 1$

$= 2$

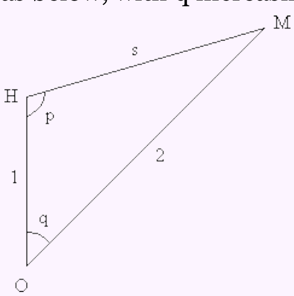
Now note that $AB:DB = BC:BA = \sqrt{2}:1$.

Since $\angle ABC$ is contained within triangles ABC and DBA, it follows that these two triangles are similar.

Hence $\angle BCA = \angle DAB = 30°$. Hence $\angle ABD = 180° - (45° + 30°) = 105°$. Therefore $\angle ABC =$ **<u>105°.</u>**

8. Calculus solution

We may regard the hour hand as fixed, and the minute hand as rotating with a constant angular speed (equal to 11/12 of its actual angular speed.) Without loss of generality, let the hour hand have length 1 and the minute hand have length 2. Let the angle between the hands be q, and the distance between the tips of the hands be s. Clearly, for s to be increasing, the diagram must be oriented as below, with q increasing, as OM rotates clockwise.



By the law of cosines (also known as the *cosine rule*), $s^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos q = 5 - 4 \cos q$.

Using implicit differentiation, $2s \cdot ds/dq = 4 \sin q$.

Hence $ds/dq = (2 \sin q)/s$.

By the law of sines (also known as the *sine rule*), $(\sin q)/s = (\sin p)/2$, where p is the angle between the hour hand and the line segment joining the tips of the hands.

Hence $ds/dq = (2 \sin p)/2 = \sin p$.

Clearly, ds/dq reaches its maximum value when sin p = 1, or, since $0° < p < 180°$, $p = \pi/2$.

By the chain rule, $ds/dt = ds/dq \cdot dq/dt$. Since dq/dt is a positive constant, ds/dt also reaches its maximum value when

$p = \pi/2$.

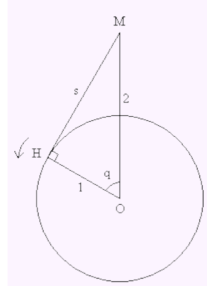
When $p = \pi/2$, OHM is a right triangle, so cos q = 1/2, and, since $0° < q < 180°$, we must have q = 60°.

Finally, we note that, as q tends to 0° from above, or to 180° from below, ds/dq tends to 0. Since s is clearly a smoothly

Geometric Solution

With a little insight (and maybe some hindsight!), a purely geometric solution is possible.

We may regard the minute hand as fixed, and the hour hand as rotating with constant angular speed. Consider the circle with center O and radius 1. As OH sweeps around this circle, it is clear that MH increases (or decreases) at its greatest rate when MH is tangent to the circle, as it is at these points that H is moving exactly away from (or towards) M. (At other points, the component of the speed of H along MH is smaller because there is also a non-zero component of the speed that is perpendicular to MH.)



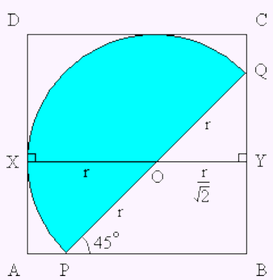
Therefore, as above, ds/dt reaches its maximum value when cos q = 1/2, which occurs at 00:**<u>10:54 6/11</u>**.

varying function of q, we conclude that ds/dq = 0 if q = 0° or q = 180°, so that the maximum occurs at q = 60°.
To find the time just after 00:00 when the angle between the hour and minute hands is 60° (1/6 revolution), consider that the angular speed of the minute hand with respect to the hour hand is 11/12 revolutions per hour. Hence q = 60° at (1/6) / (11/12) = 2/11 hours after 00:00.
Therefore, the distance between the tips of the hands is increasing at its greatest rate at 00:**10:54 6/11**.

9. It is clear that the largest semicircle will touch the sides of the square at both ends of its diameter, and will also be tangent to the perimeter.

One obvious solution is a semicircle whose diameter coincides with one side of the square.
Such a semicircle will have radius = 1/2 and area = $\pi(1/2)^2/2 = \pi/8 \approx 0.3927$.
Can we do better?

Consider a semicircle whose diameter endpoints touch two adjacent sides of the square. It is intuitively obvious that such a semicircle of maximal area will be tangent to both of the other sides of the square, but see the remarks below for a more rigorous justification.



Since the figure is symmetrical in the diagonal BD, $\angle$ QPB = 45°.

Consider the point X on AD at which the semicircle is tangent to AD. A line extended from X that is perpendicular to the tangent will be parallel to AB, and will also pass through the middle of the semicircle diameter. Let the line meet BC at Y.

OY = r cos 45° = $r/\sqrt{2}$.
Hence 1 = AB = $r + r/\sqrt{2} = r(1 + 1/\sqrt{2})$.
Thus $r = 1/(1 + 1/\sqrt{2})$.
Rationalizing the denominator, we obtain $r = 2 - \sqrt{2}$ and area = $\pi r^2/2 = \pi(3 - 2\sqrt{2})$.

Thus, the area of the largest semicircle that can be inscribed in the unit square is **$\pi(3 - 2\sqrt{2}) \approx 0.539$**.

10. Let x be the number of dollars in the check, and y be the number of cents.
Then $100y + x - 50 = 3(100x + y)$.
Therefore $97y - 299x = 50$.
A standard solution to this type of linear Diophantine equation uses Euclid's algorithm.
The steps of the Euclidean algorithm for calculating the greatest common divisor (gcd) of 97 and 299 are as follows:

$299 = 3 \times 97 + 8$

$97 = 12 \times 8 + 1$

This shows that gcd(97,299) = 1.

To solve $97y - 299x = \gcd(97,299) = 1$, we can proceed backwards, retracing the steps of the algorithm as follows:

$$1 = 97 - 8 \times 12$$
$$= 97 - (299 - 3 \times 97) \times 12$$
$$= 37 \times 97 - 12 \times 299$$

Therefore a solution to $97y - 299x = 1$ is $y = 37$, $x = 12$.

Hence a solution to $97y - 299x = 50$ is $y = 50 \times 37 = 1850$, $x = 50 \times 12 = 600$.

It can be shown that *all* integer solutions of $97y - 299x = 50$ are of the form $y = 1850 + 299k$, $x = 600 + 97k$, where k is any integer.

In this case, because x and y must be between 0 and 99, we choose $k = -6$.

This gives $y = 56$, $x = 18$.

So the check was for **$18.56.**

11.

$1/x + 1/y = -1$ (1)

$x^3 + y^3 = 4$ (2)

(1) $\Rightarrow x + y = -xy$

(2) $\Rightarrow (x + y)^3 - 3xy(x + y) = 4$

Hence $-(xy)^3 + 3(xy)^2 - 4 = 0$

By inspection, $xy = -1$ is a solution of this cubic equation.

Factorizing, we have $(xy + 1)(xy - 2)^2 = 0$.

Hence $xy = -1$, $x + y = 1$, or $xy = 2$, $x + y = -2$.

If $xy = -1$ and $x + y = 1$, then x, y are roots of the quadratic equation $u^2 - u - 1 = 0$.

(Consider the sum and product of the roots of $(u - A)(u - B) = u^2 - (A + B)u + AB = 0$.)

Hence $u = (1 \pm \sqrt{5})/2$.

If $xy = 2$ and $x + y = -2$, then x, y are roots of $u^2 + 2u + 2 = 0$.

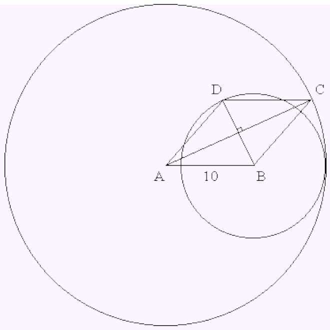
This has complex roots: $u = -1 \pm i$.

Therefore the real solutions are $x = (1 \pm \sqrt{5})/2$, $y = (1 \mp \sqrt{5})/2$.

12. Either A throws more heads than B, or A throws more tails than B, but (since A has only one extra coin) not *both*. By symmetry, these two mutually exclusive possibilities occur with equal probability.

Therefore the probability that A obtains more heads than B is ½. It is perhaps surprising that this probability is independent of the number of coins held by the players.

13. The diagram below shows the unique configuration consistent with the puzzle statement. The tangent point of the circles must lie on the extension of line AB, since their centers lie on AB.



By symmetry, the diagonals of a rhombus bisect each other and meet at right angles.
(Alternatively, for a simple vector proof, consider the dot product **AC.BD**.

AC.BD = (AB+AD).(AD–AB)

= AD.AD – AB.AB

= 0, since |AB| = |AD|

Therefore AC is perpendicular to BD.)
Let R = AC = radius of larger circle, and r = BD = radius of smaller circle.
Then, considering the four right triangles, the area of rhombus ABCD = $4 \cdot (R/2) \cdot (r/2) / 2 = Rr/2$.
Considering one of the right triangles, $(R/2)^2 + (r/2)^2 = 10^2$, from which $R^2 + r^2 = 400$.
Since the circles meet at a tangent, on AB, we have $R - r = 10$.
Hence $(R - r)^2 = R^2 + r^2 - 2Rr = 100$, and so $2Rr = 300$.
Therefore the area of the rhombus = $Rr/2 = 75$ square units.

14. The number of divisors of a natural number may be determined by writing down its prime factorization. The Fundamental Theorem of Arithmetic guarantees that the prime factorization is unique.

Let $n = p_1^{a_1} \cdot \ldots \cdot p_r^{a_r}$, where $p_1 \ldots p_r$ are prime numbers, and $a_1 \ldots a_r$ are positive integers.
Now, each divisor of n is composed of the same prime factors, where the *i*th exponent can range from 0 to a_i.
Hence there are $a_1 + 1$ choices for the first exponent, $a_2 + 1$ choices for the second, and so on.
Therefore the number of positive divisors of n is $(a_1 + 1)(a_2 + 1) \ldots (a_r + 1)$.

The unique prime factorization of 1000 is $2^3 \cdot 5^3$, which contains six prime factors.
So if n has exactly 1000 positive divisors, each $a_i + 1$ is a divisor of 1000, where i may take any value between 1 and 6.
At one extreme, when $i = 1$, $a_1 + 1 = 1000$, so $a_1 = 999$, and the smallest integer of this form is 2^{999}, a number with 301 decimal digits.
At the other extreme, when $i = 6$, $a_1 = a_2 = a_3 = 1$, and $a_4 = a_5 = a_6 = 4$.
In this latter case, the smallest integer of this form is clearly $2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 = 810{,}810{,}000$.

In fact, of all the natural numbers with exactly 1000 divisors, 810,810,000 is the smallest. A demonstration by enumeration follows. Having established this result, it will be a simple matter to find the smallest integer greater than 1 billion that has 1000 divisors.

Let $n = 2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13$.
The smallest integer that can be obtained by combining the six prime factors in various ways is:

- $5 \cdot 5 \cdot 5 \cdot 4 \cdot 2$, yielding $2^4 \cdot 3^4 \cdot 5^4 \cdot 7^3 \cdot 11 = (7^2/13)n > 3n$
- $10 \cdot 5 \cdot 5 \cdot 2 \cdot 2$, yielding $2^9 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 = (2^5/13)n > 2n$
- $25 \cdot 5 \cdot 2 \cdot 2 \cdot 2$, yielding $2^{24} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 = (2^{20}/5^3 \cdot 13)n > 500n$

- $8 \cdot 5 \cdot 5 \cdot 5$, yielding $2^7 \cdot 3^4 \cdot 5^4 \cdot 7^4 = (2^3 \cdot 7^3/11 \cdot 13)n > 10n$
- $10 \cdot 5 \cdot 5 \cdot 4$, yielding $2^9 \cdot 3^4 \cdot 5^4 \cdot 7^3 = (2^5 \cdot 7^2/11 \cdot 13)n > 10n$
- $10 \cdot 10 \cdot 5 \cdot 2$, yielding $2^9 \cdot 3^9 \cdot 5^4 \cdot 7 = (2^5 \cdot 3^4/11 \cdot 13)n > 15n$
- $20 \cdot 5 \cdot 5 \cdot 2$, yielding $2^{19} \cdot 3^4 \cdot 5^4 \cdot 7 = (2^{15}/11 \cdot 13)n > 200n$
- $25 \cdot 5 \cdot 4 \cdot 2$, yielding $2^{24} \cdot 3^4 \cdot 5^3 \cdot 7 = (2^{20}/5 \cdot 11 \cdot 13)n > 1000n$

Any other combination of the prime factors would contain a power of 2 greater than 30, which, on its own, would yield an integer greater than 1 billion.
Therefore, the smallest natural number with exactly 1000 divisors is 810,810,000.

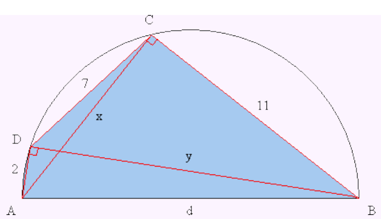
To find the smallest number greater than 1 billion with exactly 1000 divisors, we must substitute larger prime(s) in the factorization of 810,810,000.
Logically, the smallest such substitution must be either: replace $5^4 \cdot 7$ with $5 \cdot 7^4$, or replace 13 with 17.
Arithmetically, we find that $2^4 \cdot 3^4 \cdot 5 \cdot 7^4 \cdot 11 \cdot 13 = 2{,}224{,}862{,}640$, while $2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 17 = 1{,}060{,}290{,}000$.

Therefore the smallest natural number greater than 1 billion that has exactly 1000 positive divisors is 1,060,290,000.

15. Each inscribed side of the hexagon subtends an angle at the center of the circle which is independent of its position in the circle. The sides are subject to the constraint that the sum of the angles subtended at the center equals 2π.
Hence we may permute the sides of the hexagon, from {2, 2, 7, 7, 11, 11} to {2, 7, 11, 2, 7, 11}.

Since the two sets of sides, {2, 7, 11}, are congruent, each can be inscribed in a semicircle of the same radius as the original circle.

Geometric Solution

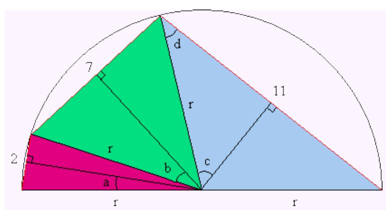


Consider cyclic quadrilateral ABCD, where AB = d is the diameter of the semicircle, BC = 11, CD = 7, and DA = 2.
Let diagonals AC = x and BD = y.
Since the angle in a semicircle is a right angle, angles ADB and ACB are right angles.

Ptolemy's Theorem states that in a cyclic quadrilateral the sum of the products of the two pairs of opposite sides equals the product of its two diagonals.
Hence $7d + 22 = xy$.

By Pythagoras' Theorem,

Trigonometric Solution



We can drop a perpendicular from the center of the circle to each of the chords, bisecting the isosceles triangles, as shown.
We have $a + b + c = \pi/2$, and $c + d = \pi/2$.
Hence $a + b = d < \pi/2$.

We also have

$$a = \sin^{-1}\left(\frac{1}{r}\right)$$

$$b = \sin^{-1}\left(\frac{7}{2r}\right)$$

$$d = \cos^{-1}\left(\frac{11}{2r}\right)$$

Taking the cosine of both sides of $a + b = d$, and using trigonometric identities $\cos(x + y) = \cos x \cos y - \sin x \sin y$, and $\sin^2 x + \cos^2 x = 1$, we get

$$\sqrt{1 - \frac{1}{r^2}}\sqrt{1 - \frac{49}{4r^2}} - \frac{7}{2r^2} = \frac{11}{2r}$$

Adding $7/2r^2$ to both sides of the equation, squaring, and

$x^2 = d^2 - 121$, and
$y^2 = d^2 - 4$

Hence $(7d + 22)^2 = (d^2 - 121)(d^2 - 4)$.
Expanding, we get $49d^2 + 308d + 484 = d^4 - 125d^2 + 484$.
Dividing by d (since $d \neq 0$) and simplifying, we obtain
$d^3 - 174d - 308 = 0$.
By the Rational Zero Theorem, a rational root of this equation must be an integer, and a factor of 308.
If we suppose the equation has an integer root, this helps us to obtain the factorization: $(d - 14)(d^2 + 14d + 22) = 0$.
The quadratic factor has negative real roots. Hence $d = 14$ is the only positive real root.

Therefore the radius of the circumscribing circle of the original hexagon is 7 units.

multiplying by $2r^3$, we obtain

$$2r^3 - 87r - 77 = 0$$

This easily factorizes, giving $(r - 7)(2r^2 + 14r + 11) = 0$.
The quadratic factor has negative real roots. Hence $r = 7$ is the only positive real root.

Therefore the radius of the circumscribing circle of the original hexagon is 7 units.

16. Let the total distance travelled downhill, on the level, and uphill, on the outbound journey, be x, y, and z, respectively.
The time taken to travel a distance s at speed v is s/v.
Hence, for the outbound journey

$$x/72 + y/63 + z/56 = 4$$

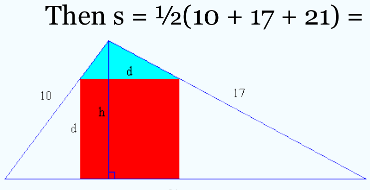
While for the return journey, which we assume to be along the same roads: $x/56 + y/63 + z/72 = 14/3$
It may at first seem that we have too little information to solve the puzzle. After all, two equations in three unknowns do not have a unique solution. However, we are not asked for the values of x, y, and z, individually;
but for the value of $x + y + z$.
Multiplying both equations by the least common multiple of denominators 56, 63, and 72, we obtain

$$7x + 8y + 9z = 4 \cdot 7 \cdot 8 \cdot 9$$
$$9x + 8y + 7z = (14/3) \cdot 7 \cdot 8 \cdot 9$$

Now it is clear that we should add the equations, yielding $16(x + y + z) = (26/3) \cdot 7 \cdot 8 \cdot 9$
Therefore $x + y + z = 273$; the distance between the two towns is 273 miles.

17. Alex and Brook cross, Alex returns
Alex and Chris cross, Chris returns
Coaches Murphy and Newlyn cross to join their Athletes
Alex and Brook cross, Alex returns
Alex and Chris cross, Chris returns
Coaches Murphy and Newlyn cross to join their Athletes
Brook and Newlyn return
Newlyn and Oakley cross, Alex returns
* All three Coaches are across
Alex and Brook cross, Coach Oakley returns
Chris and Coach Oakley cross.
Alex and Brook cross, Coach Oakley returns
Chris and Coach Oakley cross.
Done

18. By Heron's Formula, the area, A, of a triangle with sides a, b, c is given by $A = \sqrt{[s(s - a)(s - b)(s - c)]}$, where $s = ½(a + b + c)$ is the semi-perimeter of the triangle.
Then $s = ½(10 + 17 + 21) = 24$, and $A = 84$.



Now drop a perpendicular of length h onto the side of length 21.
We also have A = ½ × *base* × *perpendicular height*. Hence $A = 21h/2 = 84$, from which $h = 8$.

Notice that the triangle above the square is similar to the whole triangle. (This follows because its base, the top of the square, is parallel to the base of the whole triangle.)
Let the square have side of length d.
Considering the ratio of altitude to base in each triangle, we have $8/21 = (8 - d)/d = 8/d - 1$.
Therefore the length of the side of the square is 168/29.

19. Note that $x^3 = y^2 + 432$ is a perfect cube $\Leftrightarrow 6^3(y^2 + 432) = 216(y^2 + 432)$ is a perfect cube.
But $216(y^2 + 432) = (y + 36)^3 - (y - 36)^3$. Hence $(6x)^3 + (y - 36)^3 = (y + 36)^3$. (1)
By Fermat's Last Theorem, $a^n + b^n = c^n$ has no non-zero integer solutions for a, b and c, when $n > 2$. Here we need the result only for the case $n = 3$, which was first proved by Euler, with a gap filled by Legendre.
However, $x > 0$. Hence (1) can hold only when $y - 36 = 0$ or $y + 36 = 0$; that is, $y = \pm 36$, in which case $6x = 72$.
Therefore the only solutions are <u>***x* = 12, *y* = ±36.**</u>

20. Let the numbers be a, b, and c. Then we have

$a + b + c = 6$
$a^2 + b^2 + c^2 = 8$
$a^3 + b^3 + c^3 = 5$

We will find the (monic) cubic equation whose roots are a, b, and c.
If cubic equation $x^3 - Ax^2 + Bx - C = 0$ has roots a, b, c, then, expanding $(x - a)(x - b)(x - c)$, we find

$A = a + b + c$
$B = ab + bc + ca$
$C = abc$

Then $B = ab + bc + ca = \frac{1}{2} [(a + b + c)^2 - (a^2 + b^2 + c^2)] = 14$.
Hence a, b, c are roots of $x^3 - 6x^2 + 14x - C = 0$, and we have

$a^3 - 6a^2 + 14a - C = 0$
$b^3 - 6b^2 + 14b - C = 0$
$c^3 - 6c^2 + 14c - C = 0$

Adding, we have $(a^3 + b^3 + c^3) - 6(a^2 + b^2 + c^2) + 14(a + b + c) - 3C = 5 - 6\times8 + 14\times6 - 3C = 0$.
Hence $C = 41/3$, and $x^3 - 6x^2 + 14x - 41/3 = 0$.
Multiplying the polynomial by x, we have $x^4 - 6x^3 + 14x^2 - 41x/3 = 0$. Then

$a^4 - 6a^3 + 14a^2 - 41a/3 = 0$
$b^4 - 6b^3 + 14b^2 - 41b/3 = 0$
$c^4 - 6c^3 + 14c^2 - 41c/3 = 0$

Adding, we have $(a^4 + b^4 + c^4) - 6(a^3 + b^3 + c^3) + 14(a^2 + b^2 + c^2) - 41(a + b + c)/3 = 0$.
Hence $a^4 + b^4 + c^4 = 6\times5 - 14\times8 + (41/3)\times6 = 0$.
That is, the sum of the fourth powers of the numbers is 0.

21. Suppose the n-digit integer $s = a_1a_2a_3...a_n$ is multiplied by k when the digit a_n is transferred to the beginning of the number. (That is, $t = a_na_1a_2...a_{n-1} = ks$, where t is the resulting number.)
Note that we must have $a_1 > 0$, since s is written with no leading zeroes; and $a_n > 1$, so that when a_n is transferred to the beginning of the number, the resulting number is two or more times the original number.

Consider the infinite repeating decimals $x = 0.a_1a_2a_3...a_na_1a_2a_3...a_n...$ and $y = 0.a_na_1a_2...a_{n-1}a_na_1a_2...a_{n-1}...$, formed by repeating s and t, respectively.

We have $0.a_1a_2a_3...a_n = s/10^n$, and so $x = s/(10^n - 1)$. (This follows by considering x as the sum to infinity of a geometric series.)
Similarly, we have $0.a_na_1a_2...a_{n-1} = t/10^n = ks/10^n$, and so $y = ks/(10^n - 1)$.

Hence y = kx.

Clearly, we also have $y = a_n/10 + x/10$, or $10y = a_n + x$.
Therefore $10kx = a_n + x$, from which $x = a_n/(10k - 1)$.

We have thus reduced the problem to one of trial and error for various values of a_n and k, neither of which can be greater than 9.
We must also have $a_n \geq k$. For each value of k, we obtain the smallest value of x (and therefore of s) when $a_n = k$.

Testing each value of $k/(10k - 1)$, for $2 \leq k \leq 9$, we find the smallest s occurs for k = 4, when 4/39 yields 102564.

Hence the smallest positive integer such that when its last digit is moved to the start of the number the resulting number is larger than and is an integral multiple of the original number, is 102564.

22. Note that $2004 = 2^2 \times 501$.
Since 2 is relatively prime to 501, by Euler's Totient Theorem, $2^{\text{phi}(501)} \equiv 1 \pmod{501}$.
(Where phi(n) is Euler's totient function.)

The prime factorization of 501 is 3 × 167, so we calculate phi(501) = (3 − 1)(167 − 1) = 332.
Hence $2^{1992} = (2^{332})^6 \equiv 1^6 \equiv 1 \pmod{501}$.
Then clearly $2^{1992} = (2^2)^{996} \equiv 0 \pmod 4$.
We now combine these two results to calculate $2^{1992} \pmod{2004}$.
If $x \equiv 1 \pmod{501}$, then x = 1 + 501t, for some integer t.
Hence $1 + 501t \equiv 1 + t \equiv 0 \pmod 4$, so that $t \equiv 3 \pmod 4$.
So 1 + 501t = 1 + 501(3 + 4s) = 1504 + 2004s for some integer s.
That is, $2^{1992} \equiv 1504 \pmod{2004}$.
(The Chinese Remainder Theorem assures us that this solution is unique, mod 2004.)

Now, working modulo 2004,
$$\begin{aligned} 2^{2004} &= 2^{1992} \times 2^{12} \\ &\equiv 1504 \times 2^{12} \\ &= (1504 \times 2^2) \times 2^{10} \\ &\equiv 4 \times 1024 \\ &\equiv \mathbf{\underline{88.}} \end{aligned}$$

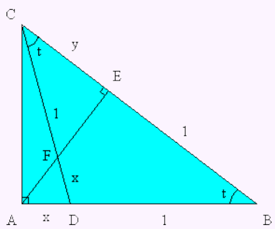
23. Consider $f(x) = \sqrt{4 + \sqrt{4 - x}}$.
Then $f(f(x)) = \sqrt{4 + \sqrt{4 - \sqrt{4 + \sqrt{4 - x}}}} = x$.
A solution to f(x) = x, if it exists, will also be a solution to f(f(x)) = x.

Solving f(x) = x

Consider, then, $f(x) = \sqrt{4 + \sqrt{4 - x}} = x$.
Let $y = \sqrt{4 - x}$. Then $y^2 = 4 - x$.
We also have $x = \sqrt{4 + y}$, from which $x^2 = 4 + y$.
Subtracting, we have $x^2 - y^2 = x + y$.
Hence $(x + y)(x - y - 1) = 0$.
Since $x \geq 0$ and $y \geq 0$, $x + y = 0 \Rightarrow x = 0$, which does not satisfy f(x) = x.
Therefore we take x − y − 1 = 0, or y = x − 1.
Substituting into $x^2 = 4 + y$, we obtain $x^2 = x + 3$, or $x^2 - x - 3 = 0$.
Rejecting the negative root, we have $\boldsymbol{x} = \dfrac{\mathbf{1+\sqrt{13}}}{\mathbf{2}}$

24. Geometric solution

Let AD = x, CE = y, and $\angle$ ABC = t. Let AE and CD meet at F.
Since $\triangle$ BCD is isosceles, $\angle$ BCD = t.
Hence $\angle$ CFE = 90° − t, and so $\angle$ DFA= 90° − t.
Since also $\angle$ FAD = $\angle$ EAB = 90° − t, $\triangle$ DFA is isosceles, and so DF = AD = x.
Hence CF = 1 − x.



Triangles ABE and CFE are similar, as each contains a right angle, and $\angle$ ABC = $\angle$ ECF.
Hence y/(1 − x) = 1/(1 + x), and so
y = (1 − x)/(1 + x)(1)

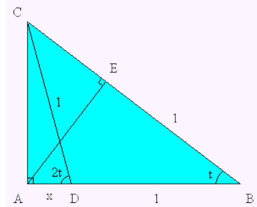
Triangles ABC and ABE are similar, as each contains a right angle, and $\angle$ ABC = $\angle$ ABE.
Hence $(1 + x)/(1 + y) = 1/(1 + x)$, and so $(1 + x)^2 = 1 + y$.

Substituting for y from (1), we obtain $(1 + x)^2 = 1 + (1 - x)/(1 + x) = 2/(1 + x)$.

Hence $(1 + x)^3 = 2$. Therefore the length of AD is $\sqrt[3]{2} - 1$

Trigonometric Solution

Let AD = x, and $\angle$ ABC = t.
Since $\triangle$ BCD is isosceles, $\angle$ BCD = t.
We also have $\angle$ BCA = 90° − t, and so $\angle$ DCA = 90° − 2t.
Hence $\angle$ ADC = 2t.



Considering triangles ABE and ADC, we obtain, respectively
cos t = $1/(1 + x)$
cos 2t = x

Applying double-angle formula $\cos 2t = 2\cos^2 t - 1$, we get
$x = 2/(1 + x)^2 - 1$

Hence $(1 + x) = 2/(1 + x)^2$, from which $(1 + x)^3 = 2$.

Therefore the length of AD is $\sqrt[3]{2} - 1$

25. It may at first seem that the sequence is not uniquely defined! However, the constraint that the sequence consists of *positive* numbers allows us to deduce the value of a_1. We will show that, if $a_1 \neq 1$, the sequence will eventually contain a negative number.

Letting $a_1 = x$, we find

$a_0 = 1 + 0x$, $a_1 = 0 + x \Rightarrow x > 0$,

$a_2 = 2 - x \Rightarrow x < 2$, $a_3 = -2 + 3x \Rightarrow x > 2/3$, (1)

$a_4 = 6 - 5x \Rightarrow x < 6/5$, $a_5 = -10 + 11x \Rightarrow x > 10/11$, ...

It seems clear that, as we calculate more and more terms, x will be "squeezed" between two fractions, both of which are part of a sequence which tends to 1 as n tends to infinity. (It would follow that x = 1.) We verify this intuition below.
Setting x = 0, to isolate the constant terms in (1), we obtain the sequence $\{b_n\}$: 1, 0, 2, −2, 6, −10, ...
We conjecture that, from $b_2 = 2$, $b_3 = -2$ onwards, the sequence alternates in sign, with $|b_{n+2}| > |b_n|$.
We prove this conjecture by mathematical induction.
Consider $b_{2n} = r$, $b_{2n+1} = -s$, where n, r, s are positive integers.
If n = 1, r = s = 2, which alternates in sign, as per the inductive hypothesis.
If n = k, $b_{2k+2} = b_{2(k+1)} = 2r + s > r > 0$, and $b_{2k+3} = b_{2(k+1)+1} = -(2r + 3s) < 0$, so that $|-(2r + 3s)| > s$.

That is, $b_{2k+2} > b_{2k} > 0$, and $b_{2k+3} < b_{2k+1} < 0$.
The result follows by induction; sequence $\{b_n\}$ alternates in sign, with $|b_{n+2}| > |b_n|$.
Setting $x = 1$, we know from the recurrence relation that $a_n = 1$, for all $n \geq 0$.
Therefore, the absolute value of the coefficient of x in sequence (1) must always differ by 1 from the absolute value of the constant term.
More specifically, we have
$a_{2n} = b_{2n} - (b_{2n} - 1)x \Rightarrow x < b_{2n}/(b_{2n} - 1)$, and
$a_{2n+1} = b_{2n+1} - (b_{2n+1} - 1)x \Rightarrow x > b_{2n+1}/(b_{2n+1} - 1)$. (Note: $b_{2n+1} < 0$.)
Since $|b_{2n}|$ and $|b_{2n+1}|$ are strictly increasing with n, the limit as n tends to infinity of both $\{b_{2n}/(b_{2n} - 1)\}$ and $\{b_{2n+1}/(b_{2n+1} - 1)\}$ is 1.
Hence <u>***x* = 1**</u>.

(For any $x \neq 1$, there exists n such that $x > b_{2n}/(b_{2n} - 1)$ or $x < b_{2n+1}/(b_{2n+1} - 1)$, and hence $a_{2n} < 0$ or $a_{2n+1} < 0$.)
Therefore $a_n = 1$, for all $n \geq 0$. Specifically, $a_{2005} = 1$.

26.

2	×	5	=	1	0
1	6	÷	8	=	2
9	=	1	5	−	6
1	8	=	9	+	9

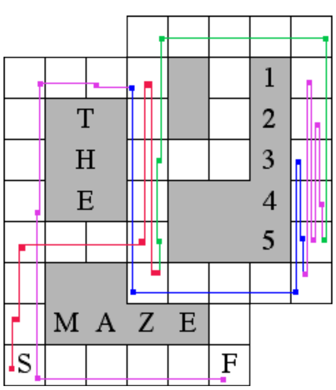
27.

```
   69
  ×46
  414
+276
 3174
```

28.

33	2	65
57	26	17
10	72	18

29.



30. $2^5 + 2^6 + 2^7 + 4^5 + 6^3 = 5^2 + 6^2 + 7^2 + 5^4 + 3^6$ and $2^3 + 2^7 + 3^6 + 5^4 + 8^2 = 3^2 + 7^2 + 6^3 + 4^5 + 2^8$

31.

$8 \times (3 \times 81 + 8) = 2008$	$8 \times 9 \times 28 - 8 = 2008$
$8 \times (4^4 - 4 - 1) = 2008$	$3 \times 8 \times 84 - 8 = 2008$
$8 \times (2^8 - 8 + 3) = 2008$	$8 \times (4^4 - 9 + 4) = 2008$
$2 \times (999 + 5) = 2008$	$8 \times (4 \times 8 \times 8 - 5) = 2008$
$7 \times 288 - 8 = 2008$	$6 \times 6 \times 56 - 8 = 2008$

32. 2469 → $4 \cdot 2469 = 9876$; $5 \cdot 2469 = 12345$

33. 19576 and 77714446555

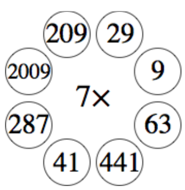
34.

$$\begin{bmatrix} 1 & 9 & 15 & 20 \\ 10 & 24 & 8 & 3 \\ 16 & 5 & 18 & 6 \\ 27 & 4 & 2 & 12 \end{bmatrix}$$

35. 891, 1170, 1290, 2931, 51070, 147970, 75914061

35. One solution: $7^2(45 - 9 + \frac{8 \cdot 3}{6} + 1)$

37.



38.

8	+	12	−	7	=	13
+		÷		+		
11	+	4	−	6	=	9
−		÷		+		
5	×	1	×	2	=	10
=		=		=		
14		3		15		

39. The scales show their combined weight to be 170 pounds, and as the lady weighs 100 pounds more than the combined weight of the dog and baby, she must have weighed exactly ***135 pounds***. As the dog weighed 60 percent less than the baby, we can readily see that the baby weighs ***25 pounds*** and the dog but ***10 pounds***.

40. The balloon travels five miles in ten minutes with the wind, but requires one hour to go back to the starting point against the wind. In 10 minutes it would travel 5/6 miles against the wind. So, in 20 minutes it would travel 5+5/6 miles in calm, without any wind i.e. It would take 20x6/35x10 minutes i.e. **34 minutes 17 and 1/7** seconds to go the whole ten miles in a calm, without any wind.

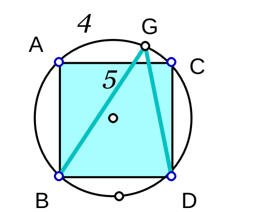
41. $3 \cdot 2^{10}$ arrangements leave an empty urn, and this number must be subtracted. However, we have now subtracted off twice the three arrangements in which two urns are empty, so we must add back 3. This gives $3^{10} - 3 \cdot 2^{10} + 3$ arrangements. (**62124**)

42.

1. 16 **L**etters of the **A**lphabet
2. 7 **D**ays of the **W**eek
3. 7 **W**onders of the **W**orld
4. 12 **S**igns of the **Z**odiac
5. 66 **B**ooks of the **B**ible
6. 52 **C**ards in a *D*eck (**W**ithout **J**okers)
7. 13 **S**tripes in the **U**.**S**. **F**lag
8. 18 **H**oles on a **G**olf **C**ourse
9. 39 **B**ooks of the Old Testament
10. 5 **D**igits on a **F**oot
11. 90 **D**egrees in a Right Angle
12. 3 **B**lind Mice (See How They Run)
13. 32 is the **T**emperature in **F**arenheit at which water **F**reezes
14. 15 **P**layers on a **R**ugby **T**eam
15. 3 **W**heels on a **T**ricycle
16. 100 **C**ents in a **D**ollar
17. 11 **P**layers on a **F**ootball (**S**occer) Team
18. 12 **M**onths in a **Y**ear
19. 13 is **U**nlucky for **S**ome
20. 8 **T**enticles on an **O**ctopus
21. 29 Days in February in a Leap Year
22. 27 Books in the New Testament
23. 354 days in a year
24. 13 Loaves in a Baker's Dozen
25. 52 Weeks in a Year
26. 9 Lives of a Cat
27. 60 Minutes in an Hour
28. 23 **P**airs of **C**hromosomes in the **H**uman **B**ody
29. 64 **S**quares on a **C**hecker(Chess) Board
30. 9 Provinces in South Africa
31. 6 Bowls to an Over in Cricket
32. 1000 Years in a Millenium
33. 15 Men on a Dead Man's Chest
34. 10 Numbers on a Telephone
35. 50 Stars on the American Flag
36. 60 Degrees in a Second
37. 16 **C**ups in a **G**allon
38. 20 **Y**ears in a **S**core
39. 4 **P**ecks in a **B**ushel
40. 6 **C**hildren in a **S**extuplets
41. 7 **S**tars in the **B**ig **D**ipper
42. **8** Legs and **8** Eyes has a **S**pider

43. Since each of the given numbers, when divided by d, has the same remainder, d divides the difference $2312 - 1417 = 895 \rightarrow 5 \cdot 179$ and $1417 - 1059 = 358 \rightarrow 2 \cdot 179$; and since 179 is prime, $d = 179$. Dividing 2312 by 179 $\rightarrow 12 \cdot 179 + 164$ and $r = 164$. $d + r = 179 + 164 =$ **343**,

44.



$AD = 5\sqrt{2}$ (diagonal of square). By the Pythagorean theorem $AD^2 = AD^2 - AG^2$ and $AD^2 = 34$. Using Ptolemy's theorem in quadrilateral $GABD \rightarrow 4 \cdot 5 + 5\sqrt{34} = PD \cdot 5\sqrt{2}$

$PD = \frac{20+5\sqrt{34}}{5\sqrt{2}} = \mathbf{2\sqrt{2} + \sqrt{17}}$

45. $\frac{\log x}{\log a}\frac{\log b}{\log x} = \frac{\log b}{\log a}$; $x > 0$ except $x \neq 1$

46. Let C be the angle across from c. $(a + b + c)(a + b - c) = (a + b)^2 - c^2 = 3ab \rightarrow a^2 + b^2 - ab = c^2$
Also $a^2 + b^2 - 2abCosC = c^2$ so $2\,abCosC = ab$ and $Cos\ C = {}^1/_2$ and $C = 60^o$.

47. If the roots are $r_1, r_2, \ldots r_{17}$, then their sum = 0, and $r_1 = 3r_1^{17} - 1$, $r_2 = 3r_2^{17} - 1$,..., $r_{17} = 3r_{17}^{17} - 1$, adding the power of the roots : $3r_1^{17} - 1 + 3r_2^{17} - 1 + \ldots + 3r_{17}^{17} - 1 = 3(r_1 + r_2 + \ldots + r_{17}) - 17 \rightarrow -17$

48. $x^2 + x + 29 = y^2$; Completing the square $\rightarrow y^2 - (x - 1/4)^2 = {}^{-115}/_4$ which yields $4y^2 - (2x + 1)^2 += 115$
$(2y + 2x + 1)(2y - 2x - 1) = 115$. Since x and y are integers $\rightarrow$ 1·115,115·1, 5· 23, 23·5) are the only possibilities.
$2y + 2x + 1 = 115 \rightarrow$ adding: $4y = 116$ and $y = 29$, $x = 28$, $2y + 2x + 1 = 23 \rightarrow 4y = 28$ and $y = 7$, $x = 4$. **(4, 7), (28, 29)**
$2y - 2x - 1 = 1$ $\qquad$ $2y - 2x - 1 = 5$

49. Ardith Jinsen, 1020 Park Lane, MSU
Crystal Miller, 1055 Park Drive, TCU
Francine Namursky, 1030 Park Road, CIT
Gilbert Evermore, 1070 Park Blvd, UU
Herbert Tomlin, 1045 Park Street, Y

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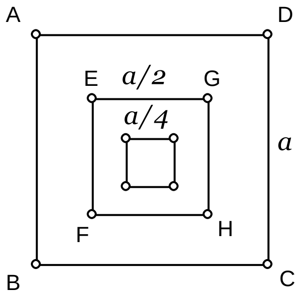	1. $P_1 = 4a$, $P_2 = 2a$, $P_3 = a$, ... $4a + 2a + a + \dots$ $S = \frac{a}{1-r} \rightarrow \frac{4a}{1/2} = 8a$ 2. $A_1 = a^2$, $A_2 = a^2/4$, $A_3 = a^2/16$, ... $a^2 + a^2/4 + a^2/16 + \dots$ $S = \frac{a^2}{1-1/4} = \frac{4a^2}{3}$