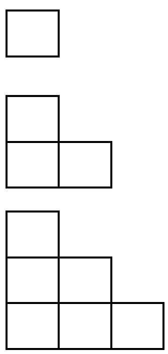
Interschool Test Part B

1. The positive root of equation: $4x^2 + ax + b = 0$ is $\sin 18^\circ$. If a and b are rational, find the ordered pair (a, b).

2. . Towns **a** and **b** are located on a straight river, **a** being 24 miles downstream from b. The river is navigable for only half the distance so that a person can row only 12 miles and must walk the other 12. Under these circumstances a person can travel from **b** to **a** in 5 hours and from **a** to **b** in 7 hours. If there were no current, the journey would require $5\frac{2}{3}$ hours. Find the person's rate of walking, rate of rowing, and the rate of the current.

3. Enclosures such as in the figure are constructed of stair patterned sets of squares. If n = the number of squares on a side, and t = the number of walls, then by actual count, for n = 1, 2, 3, 4, t = 4, 10, 18, 28.



317 walls are available for constructing such an enclosure but are not sufficient to complete it. What is the least number of additional walls needed to complete the stair pattern.

4. Jack Donavan was killed on a lonely road two miles from Trenton at 3:30 a.m. On march 17, 1933. Shorty Malone, Tony Verelli, Hank Rodgers, Joey Freiberg and Red Johnson were arrested a week later and questioned. Each of these men made four simple statements of which three were absolutely true and only one of them false. One of these five men killed Donavan.

Shorty: "I was in Chicago when Donavan was murdered. I never killed anyone. Red is the guilty man. Joey and I were pals."

Hank: "I did not kill Donavan. I never owned a revolver in my life. Red knows me. I was in Philadelphia the night of March 17."

Tony: "Hank lied when he said he never owned a revolver. The murder was committed on st. Patrick's day. Shorty was in Chicago at that time when the murder was committed. one of us five are guilty."

Joey: "I did not kill Donavan. Red has never been in Trenton. I never saw Shorty before. Hank was in Philadelphia with me the night of March 17."

Red: "I did not kill Donavan. I have never been in Trenton. I never saw Hank before now. **S**horty lied when he said I'm guilty."

Who killed Donavan?

5. On a Cartesian set of axes we call any point with integral coordinates a lattice point. A rectangle has vertices at lattice points, and its sides are parallel to the axes. Let I represent the number of interior lattice points of the rectangle ; S the number of lattice points on its sides; and A the area of the rectangle. Express A in terms of I and S.

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6. A nine digit number is formed by repeating a three digit number three times, for example, 256,256,256. Show that any such number is always divisible by 1001001.

7. Tom spent all of his money while visiting 5 stores. In each store, he spent \$1 more than half of what he had when he entered the store. How much did Tom have at the outset?

8. Find all integers (if any) a and b such that $19 - 6\sqrt{2} = a + b\sqrt{2}$.

9. Find the area included between the following graphs: $x^2 + y^2 = 4$ and $|x|$ and $|y| = 2$.

10. Find the number of ordered pairs of integers x and y such that x and y satisfy the condition:

    a) $|x| + |y| < 4$

    b) $|x| + |y| < 11$

    c) $|x| + |y| < n$

11. Find the length of the shortest altitude of a triangle whose sides are 18, 25, 27.

12. If $\sum_1^3 kx^{3-k} = 6$ solve for x

13. A circle is inscribed in an equilateral triangle and a square is inscribed in the circle. Find the ratio of the area of the triangle to the area of the square.

14. If $F(n + 1) = \frac{2F(n)+1}{2}$ for $n = 1, 2, 3, \ldots$, and $F(1) = 2$, find $F(2009)$.

15. Find the sum of all real solutions of: $(x^2 + 4x + 6)^2 = x^2 + 4x + 12$.

16. If $z = \frac{1-i\sqrt{3}}{2}$, find $\frac{3}{z-1} - \frac{1}{z}$.

17. The graphs of $x^2 + y = 9$ and $x + y = 3$ intersect in two points. If the distance between these two points is $a\sqrt{b}$, find $a + b$.

18. Opposite sides of a regular hexagon are 12 units apart. Find the perimeter of the hexagon.

19. If θ is an acute angle and $\sin 2\theta = a$, find the value of $\sin\theta$ and $\cos\theta$ in terms of a.

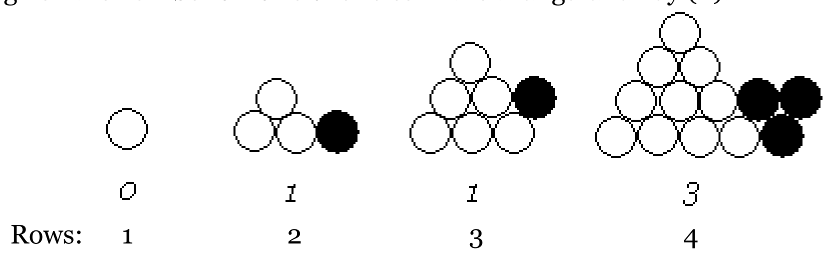
20. Solve for x: $\log_{\sqrt{2}} 4 + \log_4 \sqrt{2} - \log_2 4 - \log_4 2 = x$.

21. When the sum of the first k terms of the series: $1^2 + 2^2 + 3^2 + \ldots + n^2 + \ldots$ is subtracted from the sum of the term k terms of the series: $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1) + \ldots$. the result is 210. Find the numerical value of k.

22. If $a = ½$ and $(a + 1)(b + 1) = 2$, find the radian measure of $\tan^{-1} a + \tan^{-1} b$.

23. Find the value of $\sqrt[3]{3 + 2\sqrt{11}} + \sqrt[3]{3 - 2\sqrt{11}}$

24. Study the diagram below. Each of the triangular arrays in the figure has been transformed in to a parallelogram in a minimum number of steps. Find a formula for the minimum number of steps (M) given the number of rows of circles in the triangular array (n).

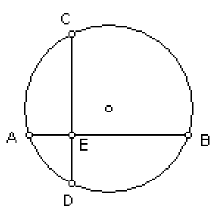


25. How many different triangles can be formed if the vertices of the triangle coincide with the vertices of a regular octagon?

26. Find the area of the largest triangle that can be inscribed in semi-circle with a radius of 10 cm.

27. Find the smallest root of $(x-3)^3 + (x+4)^3 = (2x+1)^3$.

28. In the figure AE = 2, EB = 6 and DE = 3, with AB and CD chords of the circle. Find the length of the diameter of the circle.



29. For what positive integral base is $(61_B)(51_B) = 3731_B$?

30. $F(0) = 2$, $F(1) = -2$; $F(n+1) = F(n) - F(n-1)$ for $n \geq 1$. Find $F(2009)$.

31. $\dfrac{\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}}}{\sqrt{6+4\sqrt{2}}} = a - \sqrt{b}$. Find the ordered pair (a, b).

32. The sum of the digits of a three digit number is 11. If the digits are reversed the number is increased by 99. If the sum of the first two digits is a multiple of 3, the number obtained is 8 times the third digit. Find the product of the three digits.

33. If $\tan\theta + \cot\theta = \frac{10}{3}$ and $0 < \theta < \frac{\pi}{2}$, find all possible values of $\sec\theta$.

34. On the fence are robins and crows. When 5 robins leave, there remain 2 crows for every robin. Then twenty-five crows leave, and there now three robins for every crow. Find the original number of robins.

35. Express in simplest form the real number $\left(\sqrt[3]{\sqrt{75}-\sqrt{12}}\right)^{-2}$.

36. In 1066 William of Normandy crossed the English Channel at Hastings and attacked King Harold's forces, formed as dense squares of men with interlocking shields. A contemporary history says that when Harold joined his men they "became as one mighty square, shouting the battle cries of 'Ut!', 'Olicrosse', and 'Godemite!'" If Harold's men were divided into 8 squares, and the addition of one man resulted in a new single square, how many men must there have been? (Hint: there are more than 8)

37.

15	10	?
x	14	z
y	?	?

The given magic square uses numbers 10 – 18 inclusive to obtain a sum of 42, vertically, horizontally and diagonally. Find the sum of $x + y + z$.

38. Find: $lim_{x\to 27}\frac{\sqrt{1+\sqrt[3]{x}}-2}{x-27}$.

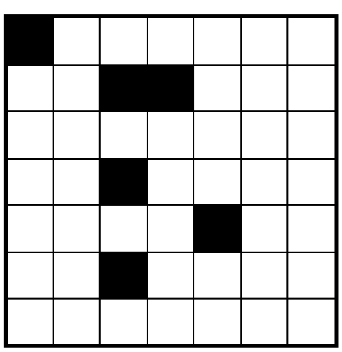
39. Find the set of points twice as far from the point (-2, 5) as from the point (3, 6).

40. Find the 2009th digit in the decimal representation of $\frac{3}{11}$. (Do not include the zero to the left of the decimal point.)

41. A person starting with \$64, makes 6 bets, winning three times and losing three times. The wins and loses come in random order, and each wager is for half the money remaining at the time the wager is made. If the change for a win equals the chance for a loss, find the final result.

42. Find the numerical value of $log_{\frac{1}{125}}25\sqrt[3]{625}$.

43.



For the given puzzle, find the starting square and draw a path moving horizontally and vertically that passes through each open square exactly once. For each segment in the path, you must go as far as possible, changing direction only when you are blocked by the grid's edge, a black square, or a square already visited.

44. "I've only met one of them," said Martha. "The guy with the beard. Which is he?"
"You figure it out yourself," Clem replied. "Two of them are married, two have blue eyes, and two are clean-shaven. The bearded one has brown eyes. Doug's wife is Ken's sister, and the bachelor has the same color eyes as Joe. They're three great guys." Who had the beard?

45. You have 12 coins, one of which is fake. The fake coin is indistinguishable from the rest except that it is either heavier or lighter, but you don't know which. Can you determine which is the fake coin **and** whether it is lighter or heavier using a balance scale and only 3 weighings? Explain.