2015 Alpha Bowl
Answers

0. 3

1. -8

2. -7

3. $-\frac{64}{3}$

4. 20,043

5. $-\frac{34}{135}$

6. 8

7. $-2e^{7}$

8. 26

9. 818

10. $-\frac{8}{3}$

11. 2053

12. $-\left(17+12\sqrt{5}\right)\pi$ or $-17\pi+12\pi\sqrt{5}$

13. -17

14. $-135+81\sqrt{3}$

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Solutions

#0

Since g is a first degree polynomial, r is a constant function. Further, $\frac{f(x)}{g(x)}=q(x)+\frac{r(x)}{g(x)}$, so $f(1)=\frac{f(1)}{3(1)-2}=\frac{f(1)}{g(1)}=q(1)+\frac{r(1)}{3(1)-2}=q(1)+r(1)=q(1)+r(2015)$, and $f(1)=6-10+1-7+7+1+5=3$.

#1

$x^{\log x^4}=\frac{x^8}{1000}\to\log x^{\log x^4}=\log x^8-\log 1000\to 4(\log x)^2-8(\log x)+3=0\to(2\log x-1)(2\log x-3)=0.$

$\log x=\frac{1}{2},\log x=\frac{3}{2}\to x=10^{\frac{1}{2}},x=10^{\frac{3}{2}}\Rightarrow\frac{10^{\frac{3}{2}}}{10^{\frac{1}{2}}}=10,A.$

$\log_5 625^{10}\to 10\log_5 5^4\Rightarrow 10(4)=40,B.$

$-3\log_8 4\to\log_8 4^{-3}\to\log_8\frac{1}{64}\Rightarrow -2,C.$

$\frac{BC}{A}=\frac{(40)(-2)}{10}=-8.$

#2

The determinant of matrix M is 1. This makes less work for us and also means that the adjoint and inverse matrices will be equal matrices. This means that A and C are the same answer.

The cofactor in row 2, column 3 is –1, A and C.

$B=$ the original element in row 2, column $3\Rightarrow -3$.

The element in row 2, column 3, of M^2 can be found by multiplying row 2 and column 3, obtaining –2, D.

$A+B+C+D=-7.$

#3

$9(x-1)^2+4(y+3)^2=36\to\frac{(x-1)^2}{4}+\frac{(y+3)^2}{9}=1\to a^2=9,b^2=4,c^2=5.$

$A=2c=2\sqrt{5}$ $B=\frac{2b^2}{a}=\frac{2(4)}{3}=\frac{8}{3}$ The center is (1, –3). One focus is $\left(1,-3+\sqrt{5}\right)$.

The directrix closer to this focus is $y=-3+\frac{9}{\sqrt{5}}$. The distance between the point and the focus is $\frac{4\sqrt{5}}{5},C$.

$ABC=\left(2\sqrt{5}\right)\left(\frac{8}{3}\right)\left(\frac{4\sqrt{5}}{5}\right)=\frac{64}{3}.$

#4

$A = 60(61) = 3660.$ $B = \frac{60(61)(121)}{6} = 73{,}810.$ $C = 2^{14} - 1 = 16{,}383.$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \rightarrow \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \ldots + \frac{1}{11} - \frac{1}{12} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12} \Rightarrow 17, D.$$

Only A and C are multiples of 3. Their sum is 20,043.

#5

$A = \left(\frac{1}{2}\right)\left(\frac{2}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{5}{9}\right) = \frac{43}{90}.$ $B = \left(\frac{1}{2}\right)\left(\frac{5}{9}\right) = \frac{5}{18}.$ $C = \left(\frac{1}{3}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{3}\right)\left(\frac{5}{9}\right) = \frac{68}{135}.$

$$A + B - C = \frac{43}{90} + \frac{5}{18} - \frac{68}{135} = \frac{68}{90} - \frac{68}{135} = \frac{204 - 136}{270} = \frac{34}{135}.$$

#6

$A = {}_4C_2(0.8)^2(0.2)^2 = \frac{96}{625}.$ $B = 1 - P(0) - P(1) = 1 - {}_4C_0(0.8)^0(0.2)^4 - {}_4C_1(0.8)^1(0.2)^3 = \frac{608}{625}.$

$$A + B = \frac{704}{625} = \frac{4^3 \cdot 11^1}{5^4} \Rightarrow 3 + 1 + 4 = 8.$$

#7

$$A = \lim_{n\to\infty} 2\left(1 + \frac{1}{n}\right)^n = 2\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2e.$$

$$B = \lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^{6n} \text{ Let } m = 2n. \lim_{\frac{m}{2}\to\infty}\left(1 + \frac{1}{m}\right)^{3m} \rightarrow \left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right]^3 = e^3.$$

$$C = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{2n} \rightarrow \left[\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n\right]^2 = e^2.$$

$$D = \lim_{n\to 0}\left(1 + 3n\right)^{\frac{1}{3n}} \text{ Let } m = \frac{1}{3n}. \lim_{\frac{1}{3m}\to 0}\left(1 + \frac{1}{m}\right)^m \rightarrow \lim_{\frac{1}{m}\to 0}\left(1 + \frac{1}{m}\right)^m \rightarrow \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = e.$$

$$ABCD = 2e^7.$$

#8

$$\begin{cases} 2^A \cdot 2^B = 8 \\ \dfrac{2^A}{2^B} = 32 \end{cases} \rightarrow \begin{cases} 2^{A+B} = 2^3 \\ 2^{A-B} = 2^5 \end{cases} \rightarrow \begin{cases} A+B=3 \\ A-B=5 \end{cases} \Rightarrow (4,-1)=(A,B)$$

$$3^{2x-1} = 2^{3x} \rightarrow 3^{2x}\cdot 3^{-1} = 2^{3x} \rightarrow \left(\frac{9}{8}\right)^x = 3 \Rightarrow \log_{\frac{9}{8}} 3 \rightarrow C=9,\ D=8$$

$$E = e^{\ln 3} + e^{\ln 2} = 3+2=5$$

$$\log_2(4x+4) - 2\log_2 x = 3 \rightarrow \log_2\left(\frac{4x+4}{x^2}\right) = 3 \rightarrow 4x+4=8x^2 \rightarrow (2x+1)(x-1)=0 \Rightarrow x=1,\ F$$

$$A+B+C+D+E+F = 4-1+9+8+5+1=26.$$

#9

$\frac{1}{7} = 0.\overline{142857}$. $\quad 6\overline{)2015}$ with quotient $335R5$ The 5th term in the pattern is 5, so the 2015th term is 5, A.

LCM is $2^3\cdot 3\cdot 5\cdot 7 = 840$, so $B = 842$.

Let the numbers in the sequence be

$a_1, a_2, a_1+a_2, a_1+2a_2, 2a_1+3a_2, 3a_1+5a_2, 5a_1+8a_2, 8a_1+13a_2, 13a_1+21a_2, 21a_1+34a_2$. This gives us $21a_1+34a_2=322$. Since $21a_1$ and 322 are divisible by 7, but 34 is not divisible by 7, a_2 must be divisible by 7. The value of a_2 must be 7; anything else would be too large. Therefore, $a_2=7, a_1=4 \Rightarrow 2a_1+3a_2=29, C$.

$A+B-C=5+842-29=818.$

#10

Using the sum-and-difference properties,

$$A = \sin\left[\operatorname{Cos}^{-1}\left(\frac{3}{5}\right)+\frac{\pi}{2}\right] = \sin\left[\operatorname{Cos}^{-1}\left(\frac{3}{5}\right)\right]\cos\left(\frac{\pi}{2}\right) + \cos\left[\operatorname{Cos}^{-1}\left(\frac{3}{5}\right)\right]\sin\left(\frac{\pi}{2}\right) = \frac{4}{5}(0)+\frac{3}{5}(1) = \frac{3}{5}.$$

$$B = \sin\left[\operatorname{Tan}^{-1}\left(-\frac{3}{4}\right)+\operatorname{Cos}^{-1}\left(-\frac{4}{5}\right)\right] = \sin\left[\operatorname{Tan}^{-1}\left(-\frac{3}{4}\right)\right]\cos\left[\operatorname{Cos}^{-1}\left(-\frac{4}{5}\right)\right] + \cos\left[\operatorname{Tan}^{-1}\left(-\frac{3}{4}\right)\right]\sin\left[\operatorname{Cos}^{-1}\left(-\frac{4}{5}\right)\right] =$$

$$\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right)+\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) \Rightarrow \frac{24}{25}.$$

Using half-angle properties:

$$C = \tan\left[\frac{1}{2}\operatorname{Arcsin}\left(\frac{15}{17}\right)\right] = \sqrt{\frac{1-\frac{8}{17}}{1+\frac{8}{17}}} = \sqrt{\frac{9}{25}} = \frac{3}{5}. \qquad \frac{B}{A\cdot C} = \frac{\frac{24}{25}}{\frac{3}{5}\cdot\frac{3}{5}} = \frac{24}{9} = \frac{8}{3}.$$

#11

$\overrightarrow{PQ} = \langle -3, 1, -7\rangle$, $\overrightarrow{PR} = \langle 0, -5, -5\rangle$. $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle -40, -15, 15\rangle$. $\overrightarrow{PQ}\times\overrightarrow{PR} = \sqrt{(-40)^2+(-15)^2+(15)^2} = \sqrt{2050}$.

Area^2=2050, A.

Using $a\cdot(b\times c)$, we get $b\times c = 18i - 36j - 18k$, so $a\cdot(b\times c) = 18-144+126 = 0$, B.

$\overrightarrow{GH} = \langle 2, 1, 1\rangle, \overrightarrow{GI} = \langle 1, -1, 2\rangle, \overrightarrow{GJ} = \langle 0, -2, 3\rangle$. $\overrightarrow{GI}\times\overrightarrow{GJ} = \langle 1, -3, -2\rangle$. $\left|\overrightarrow{GH}\cdot\left(\overrightarrow{GI}\times\overrightarrow{GJ}\right)\right| = |2-3-2| = 3$, C.

$A+B+C = 2050+0+3 = 2053.$

#12

$r = -2$ is a circle of radius 2, so its area is 4π, A. $r = 2\cos\theta$ is a circle of diameter 2, so its area is π, B.

$r = \dfrac{10}{3+2\cos\theta}$ is an ellipse. The ellipse is oriented horizontally, so the vertices are located at $(2, 0)$ and $(10, \pi)$ and the center is $(4, \pi)$. To find the minor axis endpoints, use the eccentricity, 2/3. Since $a = 6, c = 4, b = 2\sqrt{5}$. The area formula is $ab\pi \Rightarrow \left(2\sqrt{5}\right)(6)\pi = 12\pi\sqrt{5}$, C.

$\begin{cases} x = 4\cos t \\ y = 3\sin t \end{cases} \rightarrow \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. The area of this ellipse is 12π, D. The sum of the areas is $17\pi + 12\sqrt{5}\pi$.

#13

Let the three roots be $\dfrac{a}{r}, a, ar$. The product, $\dfrac{a}{r}\cdot a\cdot ar$, is a^3. From the equation itself we see that this value is 1, so $a = 1$. Substituting $x = 1$, we get $k = 7$. With synthetic division we get the depressed polynomial $2x^2 - 5x + 2 = 0 \rightarrow (2x-1)(x-2) = 0$. Therefore, our three roots are 1, 1/2, and 2.

Let $x = \sqrt[3]{\sqrt{28}+6} - \sqrt[3]{\sqrt{28}-6}$, giving $x^3 = \left(\sqrt{28}+6\right) - \left(\sqrt{28}-6\right) - 3\sqrt[3]{28-36}\left(\sqrt[3]{\sqrt{28}+6} - \sqrt[3]{\sqrt{28}-6}\right) = 12 + 6x$.
Rearranging, we get $x^3 - 6x - 12 = 0 \Rightarrow D = 0, E = -6, F = -12$.

$ABC + D + E + F = -17.$

#14

$\tan(2x) = \sqrt{3} \rightarrow 2x = 60°, 240°, 420°, 600° \Rightarrow A = 30°, B = 120°, C = 210°, D = 300°.$

For the first triangle's area, we have $\frac{1}{2}(10)(18)\sin 30° = 45$ ft². For the second triangle, we have $\frac{1}{2}(18)^2\dfrac{\sin 120°\sin 30°}{\sin 30°} = 81\sqrt{3}$ ft². For the third triangle, we have $\frac{1}{2}(10)(18) = 90$ ft². The sum of the areas is $135 + 81\sqrt{3}$ ft².