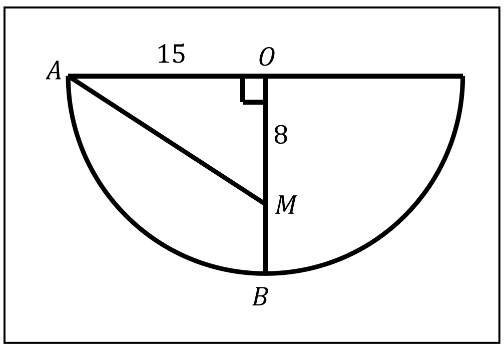
Theta Applications Solutions

1. A

The quadratics having double roots can be expressed as $x^2 + 2ax + a^2$ and $x^2 + 2bx + b^2$. Multiplying these together and gives $x^4 + 2ax^3 + 2bx^3 + a^2x^2 + b^2x^2 + 4abx^2 + 2a^2bx + 2ab^2x + a^2b^2$. We can get four new equations from this using the given equation; $a^2b^2 = q$, $2a^2b + 2ab^2 = p$, $2a + 2b = 2$, and $a^2 + b^2 + 4ab = -23$. We can simplify a few of these to get $a + b = 1$ and $(a + b)^2 + 2ab = -23$. Substituting the first into the second gives the new equation $1 + 2ab = -23$ so $ab = -12$. If $a + b = 1$ this must mean that $a = 4$ and $b = -3$ and thus $p = -24$ and $q = 144$. Reducing the fractions gives the answer -5.

2. E

Since $AB = OA$, if we cut along OA, the cone becomes a semicircle and $\overline{OA} \perp \overline{OB}$, so $AM = 17$.



Answer is 17, NOTA.

3. A

$m\angle 1 + m\angle 2 = 180 - m\angle CMU$

$$\underline{m\angle 3+m\angle 4=180-m\angle MUC}$$

$$\begin{aligned} m\angle 1+m\angle 2+m\angle 3+m\angle 4 &= 360-\left(m\angle CMU+m\angle MUC\right) \\ &= 360-\left(180-m\angle MCU\right) \\ &= 180+m\angle MCU \\ &= 180+m\angle HCP \end{aligned}$$

$$m\angle 5+m\angle 6+m\angle 7+m\angle 8=180+m\angle CPH$$

$$m\angle 9+m\angle 10+m\angle 11+m\angle 12=180+m\angle PHC$$

$$\begin{aligned} m\angle 1+m\angle 2+m\angle 3+...+m\angle 12 &= 540+\left(m\angle HCP+m\angle CPH+m\angle PHC\right) \\ &= 540+180 \\ &= 720 \end{aligned}$$

4. E

This is a linear programming problem. Let's let x=Eddybrite and y=Charlieshine. Our objective cost function is $C=3x+4y$ and our inequalities are $3x+2y\geq 9$, $4x+6y\geq 20$, $x\geq 0$, $y\geq 0$. The first inequality represents the quantities for flurrium and the second for zachium. The graph is of an unbounded region with corner points at (0, 4.5), (5, 0), and (1.4, 2.4). The minimum cost, \$13.80, is produced by (1.4, 2.4). $4(13.8+1.4+2.4)=70.4$.

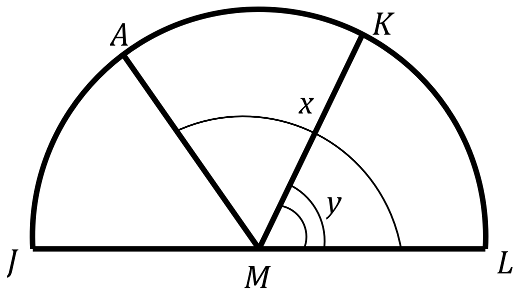
That is answer E.

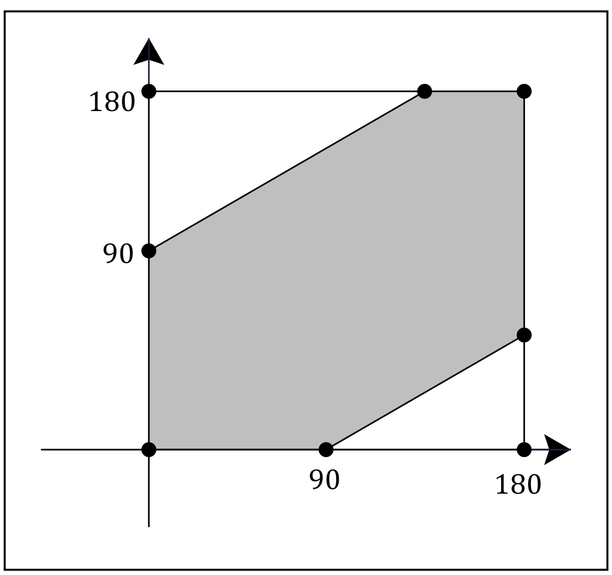
5. D

Let the roots be $\frac{d}{r}, d,$ and dr. We will now use Vieta's formulas. The product of the roots gives us $d^3=-\frac{8}{3375}\rightarrow d=-\frac{2}{15}$. Using this value of d, the sum of the roots gives us $-\frac{2}{15}\left(\frac{1}{r}+1+r\right)=-\frac{2}{15}\left(\frac{r^2+r+1}{r}\right)=\frac{7}{45}\rightarrow\frac{r^2+r+1}{r}=-\frac{7}{6}$. Cross multiply to get $6r^2+13r+6=0\rightarrow(3r+2)(2r+3)\rightarrow r=-\frac{2}{3}$ or $r=-\frac{3}{2}$. In order, where $c>f$, our roots are $\frac{1}{5},-\frac{2}{15},\frac{4}{45}$. The sum of the infinite series will be $\frac{\frac{4}{45}}{1-\frac{1}{5}}=\frac{\frac{4}{45}}{\frac{4}{5}}=\frac{1}{9}$.

This answer is D.

6. D





Let $m\angle AML = x$ and $m\angle KML = y$.
$|x - y| < 90°$, so $-90° < x - y < 90°$ or
$-90° < y - x < 90°$, so
$x - 90° < y < x + 90 \rightarrow y > x - 90$ and
$y < x + 90$. Since the semicircle measures
180°, our inequalities are bound in the first
quadrant by $x = 180$ and $y = 180$.

The area of the shaded region is
$1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$. This is D.

7. D
The men can be sat in (5 – 1)! = 24 ways and then the women can be arranged in
${}_5C_3 \times 3! = 60$ ways, for a total of 1440 ways.

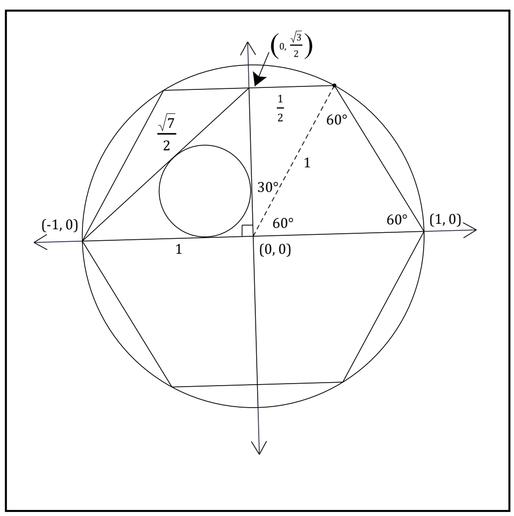
8. A

The number of distinct permutations is $\frac{11!}{3!}$, which is equivalent to $\frac{12!}{3!3!2!}$.

9. B

Paul does one-sixth of the job then Jacob does one-third more then Paula finishes. $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$, so Paula has to finish the last half of the job. Paula: $\frac{1}{2}(12)=6$ days; Jacob: $\frac{12J}{12+J}=8 \rightarrow J=24.$ $\frac{1}{3}(24)=8$ days; Paul: $\frac{24P}{24+P}=16 \rightarrow P=48.$ $\frac{1}{6}(48)=8$ days. The three workers have worked a total of 22 days. The answer is B.

10. D
Make the outer circle a unit circle on the coordinate axes.



Radius of inscribed circle: $R=\dfrac{2A}{P}=\dfrac{2\left(\frac{1}{2}\right)(1)\left(\frac{\sqrt{3}}{2}\right)}{1+\frac{\sqrt{3}}{2}+\frac{\sqrt{7}}{2}}=\dfrac{\sqrt{3}}{2+\sqrt{3}+\sqrt{7}}$.

11. C
$3x+5x+6x+7x+7x+8x=36x$
Interior angle of hexagon: (6-2)(180)=720
$36x=720\rightarrow x=20$
Smallest exterior angle: 180-20(8)=20 degrees

Exterior angle of pentagon: $15x=360\rightarrow x=24$
Largest interior angle: $180-24=156$ degrees

Difference: 136 degrees. This is C.

12. E (6)
Let the price be $\$abcd$.
$(1000a+100b+10c+d)=49(100b+10c+d)$
$1000a=4800b+480c+48d$
$1000a=48(100b+10c+d)$

$1000a=10^3a=2^3 5^3 a$ $\quad\quad$ $48=2^4\cdot 3$, so 3 is a factor of a; a must be 3, 6, or 9.
2^4 is a factor of 48 so 16 is a factor of $1000a$. Since a must be even, a is 6.
This is E (6).

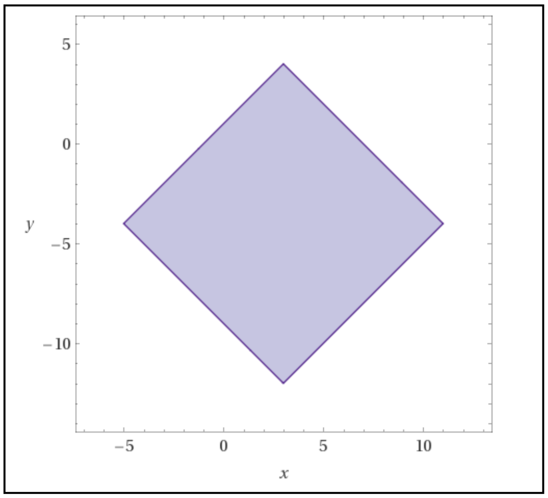
13. B
Since the constant in the denominator is 1, the numerator, 50, is the carrying capacity. The point of inflection is the point at which the growth rate begins to decrease, and it occurs at the t-value that produces half of the carrying capacity, 25.

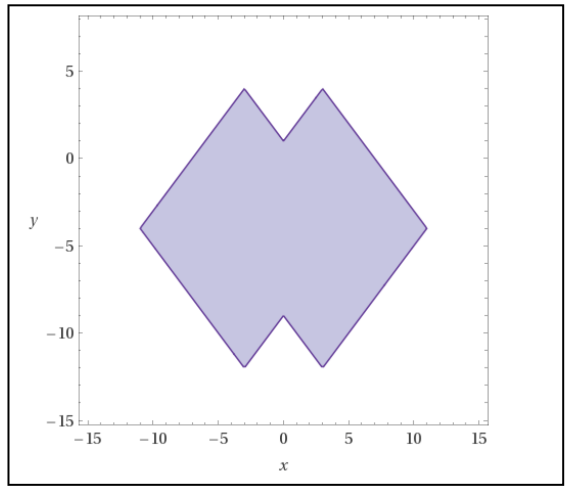
$$25=\frac{50}{1+30\cdot 10^{-2t}}\rightarrow 25+750\left(10^{-2t}\right)=50\rightarrow 10^{-2t}=\frac{1}{30}\rightarrow -2t=\log\left(\frac{1}{30}\right)=-\log 30\rightarrow$$
$$t=\frac{1}{2}(\log 10\cdot 3)=\frac{1}{2}(\log 10+\log 3)=\frac{1}{2}(1+0.4771)=\frac{1}{2}(1.4771).$$
The number of hours is $t=\dfrac{1}{2}(1.4771)(24)=17.7252\rightarrow 18$ hours.

14. D

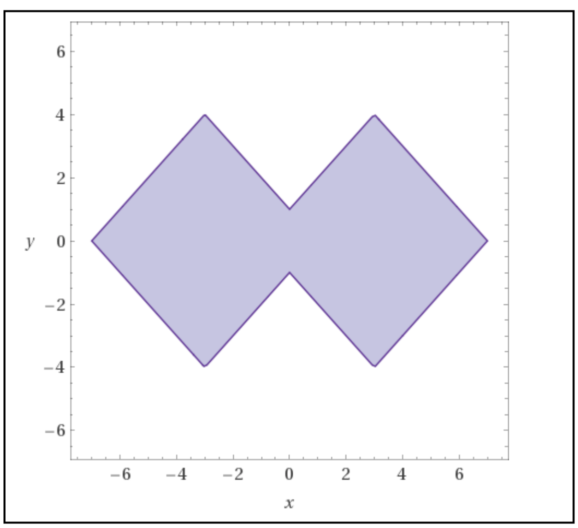
The graph of $|x-3|+|y+4|\le 8$ is a square with center (3, –4) with diagonals of length 16. The vertices are (3, 4), (11, –4), (3, –12), and (–5, –4).

The graph of $||x|-3|+|y+4|\le 8$ reflects the points with positive *x*-values over the *y*-axis (and removes all points with negative *x*-values.) The vertices are now (3, 4), (11, –4), (3, –12), (0, –9), (–3, –12), (–11, –4), (–3, 4), and (0, 1).





The graph of $||x|-3|+||y|+4|\le 8$ takes all negative *y*-value points and reflects them over the *x*-axis. The vertices are now (3, 4), (7, 0), (3, –4), (0, –1), (–3, –4), (–7, 0), (–3, 4), and (0, 1).

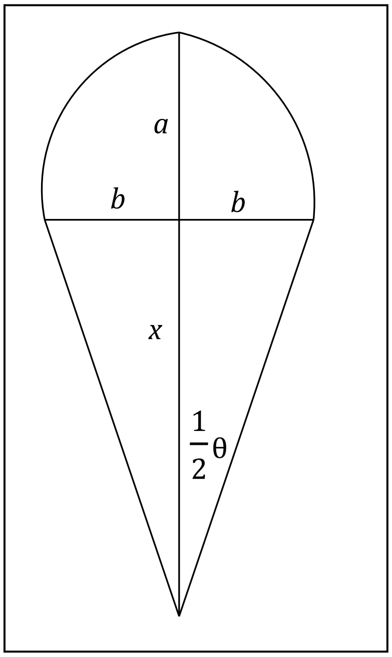


The area can be calculated by using symmetry. We will find the area of one-fourth of the polygon and multiply it by 4:

$$\pm\frac{1}{2}\begin{vmatrix} 0 & 1 \\ 3 & 4 \\ 7 & 0 \\ 0 & 0 \\ 0 & 1 \end{vmatrix} = \pm\frac{1}{2}\big((0)-(3+28)\big)=\frac{31}{2}.$$

Multiplying by 4 we get a total area of 62, so 62+8=70.

15. C



Area of semiellipse: $\frac{1}{2}ab\pi$. We know that $a = 2b$, so the area is $\frac{1}{2}(2b)b\pi = b^2\pi$.

The area of the triangle is bx.

$$bx = b^2\pi \rightarrow \frac{1}{\pi} = \frac{b}{x} \rightarrow \tan\frac{\theta}{2} = \frac{b}{x} = \frac{1}{\pi}.$$

16. B
The intersection of the absolute value function with horizontal line occurs at x=2 and x=-2. The rotation from x=0 to x=2 is symmetrical to the rotation from x=-2 to x=0. Either of the rotations creates a cone of height 2 and radius 2 so the volume of one of these is $\frac{8\pi}{3}$. Thus doubling gives the final answer of $\frac{16\pi}{3}$.

17. D
The maximum height of the ball would be $y\left(\frac{-b}{2a}\right) = y\left(\frac{3}{2}\right) = \frac{9}{4}$. Since it will cover the same distance going up as it does going down we will double the infinite geometric series formula as follows: $\frac{2a_1}{1-r} = \frac{2\frac{9}{4}}{1-\frac{5}{14}} = \frac{\frac{9}{2}}{\frac{9}{14}} = 7ft$.

18. D
Distance $= \sqrt{(x-3)^2 + (\sqrt{2x} - 0)^2} = \sqrt{x^2 - 6x + 9 + 2x} = \sqrt{x^2 - 4x + 9}$
The minimum of the distance would occur when at $x = \frac{-b}{2a} = \frac{-(-4)}{2} = 2$. The distance when x=2 would be $\sqrt{5}$.

19. E (9)
The sum can be expressed as $(x-3)^2 + (x-2)^2 + (x-1)^2 + x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 = 476$. It reduces to $7x^2 + 28 = 476$ which then gives an x of 8. 8 is the 4th largest of thee which means 9 is the final answer.

20. A

Using $P_F = P_0e^{rt}$ and the initial values of $P_F = 6, P_0 = 3$, and $t = 30$ you can solve for the rate and get $r = \frac{ln2}{30}$. Using this rate and $t = 45$ the new equation is $P_F = 3e^{45\frac{ln2}{30}}$ which simplifies to $6\sqrt{2}$.

21. E (15)

The half-life can be expressed as the equation $\frac{1}{2} = e^{5r}$. The equation for r can be solved to be $r = \frac{ln\frac{1}{2}}{5}$ and used in the equation $0.125 = e^{t\frac{ln\frac{1}{2}}{5}}$ where 0.125 represents the amount left after 87.5% of the isotope has decayed. Using the fact that $0.125 = \left(\frac{1}{2}\right)^3$ the equation can be changed to $\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{\frac{t}{5}}$ and used to solve for t to get 15.

22. B

$AB = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$. Subtracting these gives the matrix $\begin{bmatrix} 10 & 4 \\ 18.5 & 13.5 \end{bmatrix}$ whose determinant is 61.

23. E (4)

Using the general equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ you can substitute $x = 4, y = 2.4$ and $a = 5$ to give the equation $\frac{16}{25} + \frac{2.4^2}{b^2} = 1$. This can be solve for b to get 4 which is the height at the peak of the arch.

24. D

The sum can be written as $\sum_1^\infty \frac{1}{\frac{n(n+1)}{2}}$ which simplifies to $\sum_1^\infty \frac{2}{n(n+1)}$. This can be broken down to the partial fraction $2\sum_1^\infty \frac{1}{n} - \frac{1}{n+1}$. This sum's consecutive terms cancel out except for the first terms 1, which when multiplied by 2 gives the final answer of 2.

25. D

Since this is an absolute value equation of a parabola, the endpoints as well as the maximum must be considered. The maximum height of the parabola occurs at $t = 5$ and has a height of 25. The endpoints, $t = -1$ and $t = 11$ both have heights of 11 so the maximum is 25.

26. C

The equation of the second car can be modelled using the equation $y = 5t + R$ where R is the initial position of the second car. Two seconds after the first car stars driving its position is 2 and the second car has a position of $10 + R$. Since the difference between the two positions is 45.5, R must be equal to 37.5. Setting the two equations equal to each other and solving gives $5t + 37.5 = \frac{1}{2}t^2$ which can be simplified to $10t + 75 = t^2$. This quadratic only gives one positive answer of 15.

27. C
Subtracting the hour that Hansel stayed at Gretel's house gives a total of 1.5 hours spent travelling. The distance covered would have been the same there and back so the two rate/time equations can be set equal to each other, $5t_{there} = 10t_{back}$. Since the total time spent travelling was 1.5 hours the other equation to set up is $t_{there} + t_{back} = 1.5$. Solve for either of the time and plugging into the first equation gives $t_{there} = 1$ and $t_{back} = 0.5$ and a distance of 5 miles apart for the two homes.

28. A
The ladder with 8 rungs in divided into 9 equal sections and the other has 12. The difference in the two sections is $\frac{1}{3}$ of a foot. Letting x be the length of the ladder the equation can be set up as $\frac{x}{9} - \frac{x}{12} = \frac{1}{3}$. Solving for x gives an answer of 12 feet.

29. A
The "dollar" value can be changed to decimal as follows: $440_x = 4x^2 + 4x$, $1000_x = x^3$ and $340_x = 3x^2 + 4x$. The equation for the change given back will be $x^3 - (4x^2 - 4x) = 3x^2 + 4x$
The equation will simplify to $0 = x(x + 1)(x - 8)$ which only give one positive answer of 8.

30. D
Converting hours into seconds, we are looking for when the difference between a 24 hour and a 26 hour day will be 7200 seconds. The equation using seconds and days would be $7200\ sec = \left(2 \times \frac{10^{-10} sec}{day}\right) \cdot x$ days. Solving for x gives 144×10^{12} days. Converting this to years gives approximately 400 million as the answer.