Complex Numbers (Solution Keys)

1. What is $\text{Im}(z)$, the imaginary part of $z$, if $z - \bar{z} = 10i$?
   (A) 5 (B) -5 (C) 10 (D) -10 (E) NOTA

   **Solution:** $z - \bar{z} = 2\text{Im}(z)i = 10i$, so $\text{Im}(z) = 5$.
   Answer: (A)

2. Two complex numbers $z_1 = 5 + 7i$ and $z_2 = -1 + 4i$ are plotted on the complex plane. Find the complex number that divides the line segment $\overline{z_1z_2}$ by 1:2 ratio.
   (A) $2 + \frac{11}{2}i$ (B) $2 + 5i$ (C) $3 + 6i$ (D) $1 + 6i$ (E) NOTA

   **Solution:** The complex number $z$ dividing the line segment $z_1z_2$ is
   
   $z = \frac{2}{3}z_1 + \frac{1}{3}z_2 = \frac{2}{3}(5 + 7i) + \frac{1}{3}(-1 + 4i) = 3 + 6i$.
   Answer: (C)

3. Let $z, w$ be two complex numbers with $|z| = 2$ and $|w - 6 + 8i| = 5$. What is the smallest possible value of $|z - w|$?
   (A) 3 (B) 5 (C) 10 (D) 17 (E) NOTA

   **Solution:** The distance between the centers of two circles in the complex plane is 10, so the shortest distance from one circle to the other is $10 - 2 - 5 = 3$.
   Answer: (A)

4. Let $z$ be a complex number with $|z| = 10$. Which of the following is equal to $\frac{z}{25}$?
   (A) $\frac{4}{z}$ (B) $\frac{z}{4}$ (C) $4z$ (D) $\frac{1}{4z}$ (E) NOTA

   **Solution:** $z\bar{z} = 100$, so $\frac{z}{25} = \frac{4}{z}$.
   Answer: (A)

5. If $z = 1 - i$ and $w = \sqrt{3} + i$, what is the argument of $\frac{w}{z}$?
   (A) $\frac{\pi}{12}$ (B) $\frac{5\pi}{12}$ (C) $\frac{7\pi}{12}$ (D) $\frac{11\pi}{12}$ (E) NOTA

   **Solution:** $\arg(z) = -\pi/4$, $\arg(w) = \pi/6$, $\arg\left(\frac{w}{z}\right) = \arg(w) - \arg(z) = \frac{5\pi}{12}$
   Answer: (B)

6. Simplify: $(-1 + i)^{10}$
   (A) 32 (B) -32 (C) 32i (D) -32i (E) NOTA

   **Solution:** $(-1 + i)^{10} = (\sqrt{2})^{10}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)^{10} = 32\left(\cos\frac{30\pi}{4} + i \sin\frac{30\pi}{4}\right) = 32(-i)$
   Answer: (D)
7. Let \( z = a + bi \) be the complex number obtained by rotating \( 2 + 4i \) by 135°. What is \( ab \)?
   (A) 6  (B) -6  (C) 4  (D) -4  (E) NOTA

   **Solution:**
   \[
   (2 + 4i) \text{Cis}(135°) = (2 + 4i) \left( -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \right) = (1 + 2i)(-\sqrt{2} + i\sqrt{2}) = \sqrt{2}(-3 - i)
   \]
   Answer: (A)

8. Simplify: \( \frac{10i}{(1-i)(2-i)(3-i)} \)
   (A) -i  (B) i  (C) -1  (D) 1  (E) NOTA

   **Solution:**
   \[
   \frac{10i}{(1-i)(2-i)(3-i)} = \frac{10i(1+i)(2+i)(3+i)}{2 \cdot 5 \cdot 10} = \frac{(-1+3i)(1+3i)}{10} = -1
   \]
   Answer: (C)

9. What is the area of the region enclosed by a closed curve \( z \) in the complex plane if \( |z - \sqrt{3} - i\sqrt{2}| = 13 \)?
   (A) 13\(\pi\)  (B) 100\(\pi\)  (C) 144\(\pi\)  (D) 169\(\pi\)  (E) NOTA

   **Solution:** The closed curve of \( z \) satisfying |\( z - \sqrt{3} - i\sqrt{2} \)| = 13 in the complex plane is a circle centered at \((\sqrt{3}, \sqrt{2})\) with radius 13, so the area enclosed by the circle is 169\(\pi\).

   Answer: (D)

10. Find \( a + b \) if two real numbers \( a \) and \( b \) satisfy \( a(1 + 2i) + b(2 - i) = 8 + 6i \).
    (A) 6  (B) 8  (C) 12  (D) 14  (E) NOTA

   **Solution:** \( a + 2b = 8, 2a - b = 6 \), so \( a = 4, b = 2 \)

   Answer: (A)

11. Let \( z \) be a complex root of \( z^5 - 1 = 0 \). Which one of the following is equal to \( 1 + z + z^2 + \cdots + z^{2018} + z^{2019} \)?
    (A) \( i \)  (B) \(-i\)  (C) 1  (D) 0  (E) NOTA

   **Solution:**
   \[
   1 + z + z^2 + \cdots + z^{2018} + z^{2019} = (1 + z + z^2 + z^3 + z^4) + z^5(1 + z + z^2 + z^3 + z^4) + \cdots + z^{2015}(1 + z + z^2 + z^3 + z^4) = 0
   \]
   Answer: (D)

12. Let \( z \) and \( w \) be two nonzero complex numbers satisfying \( z + \overline{z} = 0 \) and \( w + \overline{w} = 0 \). What is the largest possible argument of \( \frac{z}{w} \)?
    (A) \( \frac{\pi}{4} \)  (B) \( \frac{\pi}{2} \)  (C) \( \pi \)  (D) \( \frac{3\pi}{2} \)  (E) NOTA

   **Solution:** Since both \( z \) and \( w \) are pure imaginary numbers, the largest possible angle between them is \( \pi \).
13. For a complex number $z$, if the real part of $\frac{z-1-i}{z+1+i}$ is 0, what is the distance from the origin to the point $z$ in the complex plane?

(A) $\sqrt{2}$  (B) $\frac{\pi}{2}$  (C) $\pi$  (D) $\frac{3\pi}{2}$  (E) NOTA

**Solution:** Let $z = a + bi$ where $a$ and $b$ are real. Then

$$\frac{z-1-i}{z+1+i} = \frac{a-1+(b-1)i}{a+1+(b+1)i} = \frac{a^2+b^2-2+(-2a+2b)i}{(a+1)^2+(b+1)^2} = 0,$$

so $|z| = \sqrt{a^2 + b^2} = \sqrt{2}$.

Answer: (A)

14. Consider the equation $z^6 + z^4 - z^3 + z^2 + 1 = 0$. Which of the following statement(s) is true?

a) $z^6 + z^4 - z^3 + z^2 + 1$ has three distinct factors of order 2.

b) There are exactly 6 distinct roots over complex number system, which are three pairs of complex conjugates.

c) The sum of the imaginary parts of all roots is positive.

(A) b  (B) b and c  (C) a and b  (D) a  (E) NOTA

**Solution:** Since $z^6 + z^4 - z^3 + z^2 + 1 = (z^2 + z + 1)(z^4 - z^3 + z^2 - z + 1) = \frac{(z^3-1)(z^5+1)}{(z-1)(z+1)}$, the roots of $z^6 + z^4 - z^3 + z^2 + 1 = 0$ are exactly six non real complex numbers out of the three roots of $z^3 = 1$ and five roots of $z^5 = -1$. Therefore, there are three pairs of complex conjugate roots and the sum of the imaginary parts of them is 0.

Answer: (A)

15. Given three vertices $4 + i$, $-1 - 2i$, $2 + 7i$ of a parallelogram, which one of the following complex numbers can be the fourth vertex?

(A) $1 + i$  (B) $7 + 10i$  (C) $-4 - 5i$  (D) $-5 - 4i$  (E) NOTA

**Solution:** By inspection, the midpoint of $7 + 10i$ and $-1 - 2i$ coincide the midpoint of $4 + i$ and $2 + 7i$.

Answer: (B)

16. Let $m$ and $n$ be the smallest positive integers such that $(1 + i\sqrt{3})^m = (1 - i)^n$. What is the value of $m + n$?

(A) 12  (B) 24  (C) 36  (D) 48  (E) NOTA

**Solution:** $(1 + i\sqrt{3})^m = 2^m \left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)^m = 2^m \left(\cos\frac{m\pi}{3} + i \sin\frac{m\pi}{3}\right)$ and
\[(1 - i)^n = (\sqrt{2})^n \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right)^n = (\sqrt{2})^n \left( \cos \frac{7n\pi}{4} + i \sin \frac{7n\pi}{4} \right).\] Thus, \(n = 2m\) and \(\frac{m\pi}{3} = \frac{7n\pi}{4} + 2\pi k\) for any integer \(k\).

Since \(k = \frac{19m}{12}\) must be an integer, the smallest positive choice of \(m\) is 12, and hence \(n = 24\).

Answer: (C)

17. If \(2 + i\) is a root of \(f(x) = x^3 + ax^2 + bx - 20\) where \(a\) and \(b\) are real numbers, what is the value of \(a + b\)?

(A) -5  (B) 5  (C) -13  (D) 13  (E) NOTA

**Solution:** The other solutions are \(2 - i\) and \(4\), so \(a = -(2 - i + 2 + i + 4) = -8\) and \(b = (2 - i)(2 + i) + (2 - i)(4) + (2 + i)(4) = 21\).

Answer: (D)

18. Let \(z_1\) and \(z_2\) be two solutions of the quadratic equation \(x^2 - 2x + 2 = 0\). If \(z\) is a complex number such that \(\Delta zz_1 z_2\) forms an equilateral triangle, what is the sum of all possible values of \(z\)?

(A) 2  (B) 0  (C) \(2\sqrt{3}\)  (D) \(\sqrt{3}\)  (E) NOTA

**Solution:** \(z_1\) and \(z_2\) are \(1 + i\) and \(1 - i\), and the distance between them is 2. Thus \(z\) is either \(1 + \sqrt{3}\) or \(1 - \sqrt{3}\).

Answer (A)

19. Let \(z_1, z_2, z_3\) be three complex numbers with \(|z_1 - z_2| = 7\) and \(|z_2 - z_3| = 4\). If we let \(M\) and \(m\) be the maximum distance and the minimum distance between \(z_1\) and \(z_3\), respectively, what is \(M + m\)?

(A) 11  (B) 12  (C) 13  (D) 14  (E) NOTA

**Solution:** The lotus of the points \(z_2\) is a circle centered at \(z_1\) with radius 7, and \(z_3\) lies on circles centered at \(z_2\) with radius 4. The maximum and the minimum distances between \(z_1\) and \(z_3\) occur when \(z_1, z_2, z_3\) are collinear. \(M = 11\) and \(m = 3\).

Answer: (D)

20. Let \(z_1, z_2, z_3, z_4, z_5\) be 5 vertices on the unit circle form a regular pentagon. What is the product of the distances from one vertex to each of the other 4 vertices?

(A) 4  (B) 6  (C) 8  (D) 10  (E) NOTA

**Solution:** \(z_1, z_2, \cdots, z_5\) are the roots of \(z^5 - 1 = 0\), so \(z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1) = (z - z_1)(z - z_2) \cdots (z - z_5) = 0\) and \(z_1 = 1\). Thus, \((1 - z_2)(1 - z_3)(1 - z_4)(1 - z_5) = 5\).

Answer: (E)
21. For how many number of real numbers \( x \) is \((x + i)^{4}\) real?
   (A) 1  (B) 2  (C) 3  (D) 4  (E) NOTA

   **Solution:** \((x + i)^{4} = x^{4} - 6x^{2} + 1 + i(4x^{3} - 4x) . 4x^{3} - 4x = 0 \) if and only if \((x + i)^{4}\) is real.

   Answer: (C)

22. Let \( \omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \). Which one of the following is NOT true?
   (A) \( \omega^{2} = \overline{\omega} \)  (B) \( \omega^{3} = -1 \)  (C) \( \overline{\omega} = 1/\omega \)  (D) \( \omega^{2} = -\omega - 1 \)  (E) NOTA

   **Solution:** \( \omega^{3} = 1 \), so \( \omega^{2} + \omega + 1 = 0 \), and hence \( \overline{\omega} \omega = 1 \) and \( \omega^{2} = \overline{\omega} \).

   Answer: (B)

23. Which one of following best describes the graph of the equation \(|z| + |z - 2 + 4i| = 3\) in the complex plane?
   (A) A line  (B) A circle  (C) An ellipse  (D) A parabola  (E) NOTA

   **Solution:** Let \( z = x + yi \), then \(|z| + |z - 2 + 4i| = \sqrt{x^{2} + y^{2}} + \sqrt{(x - 2)^{2} + (y + 4)^{2}} = 3\).

   This yields an ellipse.

   Answer: (C)  - Changed to E

24. Let \( z \) be a complex number and \( \overline{z} \) be the complex conjugate of \( z \). If both \( \frac{z}{10} \) and \( \frac{10}{z} \) have real and imaginary parts between 0 and 1, inclusive, what is the smallest value of \(|z|\)?
   (A) \( \sqrt{2} \)  (B) \( 5\sqrt{2} \)  (C) 10  (D) 25  (E) NOTA

   **Solution:** Let \( z = x + yi \), then \( \frac{z}{10} = \frac{x}{10} + i \frac{y}{10} \), and \( \frac{10}{z} = \frac{10x}{x^{2}+y^{2}} + i \frac{10y}{x^{2}+y^{2}} \).

   Since \( 0 \leq \frac{x}{10} \leq 1, 0 \leq \frac{y}{10} \leq 1, 0 \leq \frac{10x}{x^{2}+y^{2}} \leq 1, \) and \( 0 \leq \frac{10y}{x^{2}+y^{2}} \leq 1, \) we have \( 0 \leq x \leq 10, 0 \leq y \leq 10, \)

   \((x - 5)^2 + y^2 \geq 25, \) and \( x^2 + (y - 5)^2 \geq 25. \)

   Thus, the smallest value of \(|z|\) occurs when \( x = 5 \) and \( y = 5 \) in the area and

   \(|z| = \sqrt{5^2 + 5^2} = 5\sqrt{2}. \)

   Answer: (B)

25. When \( i - \frac{1}{i} \) is a root of a quadratic equation with real coefficients, what is the other root of the same equation?
   (A) \( i + \frac{1}{i} \)  (B) \( 2i \)  (C) \( -\frac{2}{i} \)  (D) \( \frac{2}{i} \)  (E) NOTA

   **Solution:** Since \( i - \frac{1}{i} = 2i \), its complex conjugate \(-2i\) is also a root, so \(-2i = \frac{2i}{-1} = \frac{2i}{i^2} = \frac{2}{i}\) is a solution.
26. If \( f(n) = \left( \frac{1+i}{1-i} \right)^n + \left( \frac{1-i}{1+i} \right)^n \), find the sum \( \sum_{n=1}^{2018} f(n) \).

(A) 2       (B) -2       (C) 2i       (D) -2i       (E) NOTA

**Solution:** \( f(n) = (i)^n + (-i)^n \). \( \sum_{n=1}^{2018} f(n) = f(2) + f(4) + \cdots + f(2018) = 2((i)^2 + (i)^4 + (i)^6 + \cdots + (i)^{2018}) = -2 \)

Answer: (B)

27. Assume that \( z_1, z_2, z_3 \) are complex numbers with \( \frac{z_2 - z_1}{z_3 - z_1} = \sqrt{3} + i \). If the area of the triangle \( \Delta z_1z_2z_3 \) is equal to 18, what is \( |z_3 - z_1| \)?

(A) 4       (B) 5       (C) 6       (D) 7       (E) OTA

**Solution:** \( \arg \frac{z_2 - z_1}{z_3 - z_1} = \frac{\pi}{6} \) and \( |\frac{z_2 - z_1}{z_3 - z_1}| = 2 \). Since \( 18 = \frac{1}{2} |z_2 - z_1| |z_3 - z_1| \sin \frac{\pi}{6}, |z_3 - z_1| = 6 \).

Answer: (C)

28. Let \( z \) and \( w \) be two nonzero complex numbers satisfying \( z^6 + z^3 + 1 = 0 \) and \( w^6 - w^3 + 1 = 0 \). How many distinct complex numbers of \( zw \) are possible?

(A) 6       (B) 9       (C) 12       (D) 18       (E) NOTA

**Solution:** Since \( z^6 + z^3 + 1 = \frac{(z^3-1)(z^6+z^3+1)}{(z^3-1)} = \frac{z^9-1}{z^3-1} = 0 \), only the six roots out of the nine roots of \( z^9 - 1 = 0 \), which do not satisfy \( z^3 - 1 = 0 \), are the roots of \( z^6 + z^3 + 1 = 0 \). The possible \( z \) is in the form of \( z = \cos \alpha + i \sin \alpha \) where \( \alpha = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9} \). Similarly, there are six roots of \( w^6 - w^3 + 1 = \frac{(w^3-1)(w^6-w^3+1)}{(w^3-1)} = \frac{w^9+1}{w^3+1} = 0 \) with the form of \( w = \cos \beta + i \sin \beta \) where \( \beta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9} \). Therefore, \( zw = \cos(\alpha + \beta) + i \sin(\alpha + \beta) \) where \( \alpha + \beta = \frac{\pi}{9}, \frac{3\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \cdots, \frac{17\pi}{9} \).

Answer: (B)

29. Let \( z_1 \) be the root of \( z^5 = 1 \) with the smallest positive imaginary part. Let \( z_2 \) be the root of \( z^7 = 1 \) with the smallest positive imaginary part. What is the argument of \( z_1z_2 \)?

(A) \( \frac{2\pi}{35} \)       (B) \( \frac{12\pi}{35} \)       (C) \( \frac{24\pi}{35} \)       (D) \( \frac{58\pi}{35} \)       (E) NOTA

**Solution:** \( \arg(z_1) = \frac{4\pi}{5} \) and \( \arg(z_2) = \frac{6\pi}{7} \), so \( \arg(z_1z_2) = \frac{58\pi}{35} \).

Answer: (D)

30. Let \( x \) and \( y \) be two nonzero complex numbers satisfying \( x^2 + xy + y^2 = 0 \). What is the value of \( \left( \frac{x}{x+y} \right)^{100} + \left( \frac{y}{x+y} \right)^{100} \)?

(A) 0       (B) -1       (C) 1       (D) 2       (E) NOTA
Solution: $x^2 + xy + y^2 = 0$ yields $\left(\frac{x}{y}\right)^2 + \frac{x}{y} + 1 = 0$. Let $w = \frac{x}{y}$, then $w^2 + w + 1 = 0$, and hence $w^3 = 1$. Now $\left(\frac{x}{x+y}\right)^{100} + \left(\frac{y}{x+y}\right)^{100} = \frac{x^{100} + y^{100}}{(x+y)^{100}} = \frac{w^{100} + 1}{(w+1)^{100}} = \frac{w+1}{(-w^2)^{100}} = \frac{-w^2}{w^2} = \frac{-w^2}{w^2} = -1.$

Answer (B)