Complex Numbers (Solution Keys)

1. What is $Im(z)$, the imaginary part of z, if $z - \bar{z} = 10i$?
(A) 5 (B) -5 (C) 10 (D) -10 (E) NOTA

Solution: $z - \bar{z} = 2\ Im(z)i = 10i$, so $Im(z) = 5$.

Answer: (A)

2. Two complex numbers $z_1 = 5 + 7i$ and $z_2 = -1 + 4i$ are plotted on the complex plane. Find the complex number that divides the line segment $\overline{z_1 z_2}$ by 1:2 ratio.
(A) $2 + \frac{11}{2}i$ (B) $2 + 5i$ (C) $3 + 6i$ (D) $1 + 6i$ (E) NOTA

Solution: The complex number z dividing the line segment $\overline{z_1 z_2}$ is

$z = \frac{2}{3}z_1 + \frac{1}{3}z_2 = \frac{2}{3}(5 + 7i) + \frac{1}{3}(-1 + 4i) = 3 + 6i.$

Answer: (C)

3. Let z, w be two complex numbers with $|z| = 2$ and $|w - 6 + 8i| = 5$. What is the smallest possible value of $|z - w|$?
(A) 3 (B) 5 (C) 10 (D) 17 (E) NOTA

Solution: The distance between the centers of two circles in the complex plane is 10, so the shortest distance from one circle to the other is $10 - 2 - 5 = 3$.

Answer: (A)

4. Let z be a complex number with $|z| = 10$. Which of the following is equal to $\frac{z}{25}$?
(A) $\frac{4}{\bar{z}}$ (B) $\frac{\bar{z}}{4}$ (C) $4z$ (D) $\frac{1}{4\bar{z}}$ (E) NOTA

Solution: $z\bar{z} = 100$, so $\frac{z}{25} = \frac{4}{\bar{z}}$.

Answer: (A)

5. If $z = 1 - i$ and $= \sqrt{3} + i$, what is the argument of $\frac{w}{z}$?
(A) $\frac{\pi}{12}$ (B) $\frac{5\pi}{12}$ (C) $\frac{7\pi}{12}$ (D) $\frac{11\pi}{12}$ (E) NOTA

Solution: $\arg(\mathrm{z}) = -\pi/4$, $\arg(w) = \pi/6$, $\arg\left(\frac{w}{z}\right) = \arg(w) - \arg(z) = \frac{5\pi}{12}$

Answer: (B)

6. Simplify: $(-1 + i)^{10}$
(A) 32 (B) -32 (C) $32i$ (D) $-32i$ (E) NOTA

Solution: $(-1 + i)^{10} = \left(\sqrt{2}\right)^{10}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{10} = 32\left(\cos\frac{30\pi}{4} + i\sin\frac{30\pi}{4}\right) = 32(-i)$

Answer: (D)

7. Let $z = a + bi$ be the complex number obtained by rotating $2 + 4i$ by $135°$. What is ab?
(A) 6 (B) -6 (C) 4 (D) -4 (E) NOTA

Solution: $(2+4i)\text{Cis}(135°) = (2+4i)\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right) = (1+2i)\left(-\sqrt{2} + i\sqrt{2}\right) = \sqrt{2}(-3-i)$
Answer: (A)

8. Simplify: $\frac{10i}{(1-i)(2-i)(3-i)}$
(A) $-i$ (B) i (C) -1 (D) 1 (E) NOTA

Solution: $\frac{10i}{(1-i)(2-i)(3-i)} = \frac{10i(1+i)(2+i)(3+i)}{2\cdot 5\cdot 10} = \frac{(-1+3i)(1+3i)}{10} = -1$

Answer: (C)

9. What is the area of the region enclosed by a closed curve z in the complex plane if $\left|z - \sqrt{3} - i\sqrt{2}\right| = 13$?
(A) 13π (B) 100π (C) 144π (D) 169π (E) NOTA

Solution: The closed curve of z satisfying $\left|z - \sqrt{3} - i\sqrt{2}\right| = 13$ in the complex plane is a circle centered at $(\sqrt{3}, \sqrt{2})$ with radius 13, so the area enclosed by the circle is 169π.

Answer: (D)

10. Find $a + b$ if two real numbers a and b satisfy $a(1+2i) + b(2-i) = 8 + 6i$.
(A) 6 (B) 8 (C) 12 (D) 14 (E) NOTA

Solution: $a + 2b = 8, 2a - b = 6$, so $a = 4, b = 2$

Answer: (A)

11. Let z be a complex root of $z^5 - 1 = 0$. Which one of the following is equal to $1 + z + z^2 + \cdots + z^{2018} + z^{2019}$?
(A) i (B) $-i$ (C) 1 (D) 0 (E) NOTA

Solution: $1 + z + z^2 + \cdots + z^{2018} + z^{2019} = (1 + z + z^2 + z^3 + z^4) + z^5(1 + z + z^2 + z^3 + z^4) + \cdots + z^{2015}(1 + z + z^2 + z^3 + z^4) = 0$

Answer: (D)

12. Let z and w be two nonzero complex numbers satisfying $z + \bar{z} = 0$ and $w + \bar{w} = 0$. What is the largest possible argument of $\frac{z}{w}$?
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) π (D) $\frac{3\pi}{2}$ (E) NOTA

Solution: Since both z and w are pure imaginary numbers, the largest possible angle between them is π.

Answer: (C)

13. For a complex number z, if the real part of $\frac{z-1-i}{z+1+i}$ is 0, what is the distance from the origin to the point z in the complex plane?

(A) $\sqrt{2}$ (B) $\frac{\pi}{2}$ (C) π (D) $\frac{3\pi}{2}$ (E) NOTA

Solution: Let $z = a + bi$ where a and b are real. Then $\frac{z-1-i}{z+1+i} = \frac{a-1+(b-1)i}{a+1+(b+1)i} = \frac{a^2+b^2-2+(-2a+2b)i}{(a+1)^2+(b+1)^2} = 0$, so $|z| = \sqrt{a^2+b^2} = \sqrt{2}$.

Answer: (A)

14. Consider the equation $z^6 + z^4 - z^3 + z^2 + 1 = 0$. Which of the following statement(s) is true?
   a) $z^6 + z^4 - z^3 + z^2 + 1$ has three distinct factors of order 2.
   b) There are exactly 6 distinct roots over complex number system, which are three pairs of complex conjugates.
   c) The sum of the imaginary parts of all roots is positive.

(A) b (B) b and c (C) a and b (D) a (E) NOTA

Solution: Since $z^6 + z^4 - z^3 + z^2 + 1 = (z^2 + z + 1)(z^4 - z^3 + z^2 - z + 1) = \frac{(z^3-1)(z^5+1)}{(z-1)(z+1)}$, the roots of $z^6 + z^4 - z^3 + z^2 + 1 = 0$ are exactly six non real complex numbers out of the three roots of $z^3 = 1$ and five roots of $z^5 = -1$. Therefore, there are three pairs of complex conjugate roots and the sum of the imaginary parts of them is 0.

Answer: (A)

15. Given three vertices $4 + i, -1 - 2i, 2 + 7i$ of a parallelogram, which one of the following complex numbers can be the fourth vertex?

(A) $1 + i$ (B) $7 + 10i$ (C) $-4 - 5i$ (D) $-5 - 4i$ (E) NOTA

Solution: By inspection, the midpoint of $7 + 10i$ and $-1 - 2i$ coincide the midpoint of $4 + i$ and $2 + 7i$.

Answer: (B)

16. Let m and n be the smallest positive integers such that $\left(1 + i\sqrt{3}\right)^m = (1 - i)^n$. What is the value of $+n$?

(A) 12 (B) 24 (C) 36 (D) 48 (E) NOTA

Solution: $\left(1 + i\sqrt{3}\right)^m = 2^m\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^m = 2^m\left(\cos\frac{m\pi}{3} + i\sin\frac{m\pi}{3}\right)$ and

$(1-i)^n = (\sqrt{2})^n \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)^n = (\sqrt{2})^n \left(\cos\frac{7n\pi}{4} + i\sin\frac{7n\pi}{4}\right)$. Thus, $n = 2m$ and

$\frac{m\pi}{3} = \frac{7n\pi}{4} + 2\pi k$ for any integer k.

Since $k = \frac{19m}{12}$ must be an integer, the smallest positive choice of m is 12, and hence $n = 24$.

Answer: (C)

17. If $2+i$ is a root of $f(x) = x^3 + ax^2 + bx - 20$ where a and b are real numbers, what is the value of $a+b$?
(A) -5 (B) 5 (C) -13 (D) 13 (E) NOTA

Solution: The other solutions are $2-i$ and 4, so $a = -(2-i+2+i+4) = -8$ and

$b = (2-i)(2+i) + (2-i)(4) + (2+i)(4) = 21.$

Answer: (D)

18. Let z_1 and z_2 be two solutions of the quadratic equation $x^2 - 2x + 2 = 0$. If z is a complex number such that $\Delta z z_1 z_2$ forms an equilateral triangle, what is the sum of all possible values of z?
(A) 2 (B) 0 (C) $2\sqrt{3}$ (D) $\sqrt{3}$ (E) NOTA

Solution: z_1 and z_2 are $1+i$ and $1-i$, and the distance between them is 2. Thus z is either $1+\sqrt{3}$ or $1-\sqrt{3}$.

Answer (A)

19. Let z_1, z_2, z_3 be three complex numbers with $|z_1 - z_2| = 7$ and $|z_2 - z_3| = 4$. If we let M and m be the maximum distance and the minimum distance between z_1 and z_3, respectively, what is $M+m$?
(A) 11 (B) 12 (C) 13 (D) 14 (E) NOTA

Solution: The lotus of the points z_2 is a circle centered at z_1 with radius 7, and z_3 lies on circles centered at z_2 with radius 4. The maximum and the minimum distances between z_1 and z_3 occur when z_1, z_2, z_3 are collinear. $M = 11$ and $m = 3$.

Answer: (D)

20. Let z_1, z_2, z_3, z_4, z_5 be 5 vertices on the unit circle form a regular pentagon. What is the product of the distances from one vertex to each of the other 4 vertices?
(A) 4 (B) 6 (C) 8 (D) 10 (E) NOTA

Solution: $z_1, z_2, \cdots, z_5$ are the roots of $z^5 - 1 = 0$, so $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1) = (z - z_1)(z - z_2)\cdots(z - z_5) = 0$ and $z_1 = 1$. Thus, $(1-z_2)(1-z_3)(1-z_4)(1-z_5) = 5$.

Answer: (E)

21. For how many number of real numbers x is $(x+i)^4$ real?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

Solution: $(x+i)^4 = x^4 - 6x^2 + 1 + i(4x^3 - 4x)$. $4x^3 - 4x = 0$ if and only if $(x+i)^4$ is real.

Answer: (C)

22. Let $= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Which one of the following is NOT true?

(A) $w^2 = \overline{w}$ (B) $w^3 = -1$ (C) $\overline{w} = 1/w$ (D) $w^2 = -w - 1$ (E) NOTA

Solution: $w^3 = 1$, so $w^2 + w + 1 = 0$, and hence $w\overline{w} = 1$ and $w^2 = \overline{w}$.

Answer: (B)

23. Which one of following best describes the graph of the equation $|z| + |z - 2 + 4i| = 3$ in the complex plane?

(A) A line (B) A circle (C) An ellipse (D) A parabola (E) NOTA

Solution: Let $z = x + yi$, then $|z| + |z - 2 + 4i| = \sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + (y+4)^2} = 3.$ This yields an ellipse.

Answer: (C) - Changed to E

24. Let z be a complex number and $\bar{z}$ be the complex conjugate of z. If both $\frac{z}{10}$ and $\frac{10}{\bar{z}}$ have real and imaginary parts between 0 and 1, inclusive, what is the smallest value of $|z|$?

(A) $\sqrt{2}$ (B) $5\sqrt{2}$ (C) 10 (D) 25 (E) NOTA

Solution: Let $z = x + yi$, then $\frac{z}{10} = \frac{x}{10} + i\frac{y}{10}$, and $\frac{10}{\bar{z}} = \frac{10x}{x^2+y^2} + i\frac{10y}{x^2+y^2}$. Since $0 \le \frac{x}{10} \le 1$,

$0 \le \frac{y}{10} \le 1$, $0 \le \frac{10x}{x^2+y^2} \le 1$, and $0 \le \frac{10y}{x^2+y^2} \le 1$, we have $0 \le x \le 10$, $0 \le y \le 10$,

$(x-5)^2 + y^2 \ge 25$, and $x^2 + (y-5)^2 \ge 25$.

Thus, the smallest value of $|z|$ occurs when $x = 5$ and $y = 5$ in the area and

$|z| = \sqrt{5^2 + 5^2} = 5\sqrt{2}$.

Answer: (B)

25. When $i - \frac{1}{i}$ is a root of a quadratic equation with real coefficients, what is the other root of the same equation?

(A) $i + \frac{1}{i}$ (B) $2i$ (C) $-\frac{2}{i}$ (D) $\frac{2}{i}$ (E) NOTA

Solution: Since $i - \frac{1}{i} = 2i$, its complex conjugate $-2i$ is also a root, so $-2i = \frac{2i}{-1} = \frac{2i}{i^2} = \frac{2}{i}$ is a solution.

Answer: (D)

26. If $f(n) = \left(\frac{1+i}{1-i}\right)^n + \left(\frac{1-i}{1+i}\right)^n$, find the sum $\sum_{n=1}^{2018} f(n)$.

(A) 2 (B) -2 (C) $2i$ (D) $-2i$ (E) NOTA

Solution: $f\ (n) = (i)^n + \ (-i)^n$. $\sum_{n=1}^{2018} f(n) = f(2) + f(4) + \cdots + f(2018) = 2((i)^2 + (i)^4 + (i)^6 + \cdots + (i)^{2018}) = -2$

Answer: (B)

27. Assume that z_1, z_2, z_3 are complex numbers with $\frac{z_2-z_1}{z_3-z_1} = \sqrt{3} + i$. If the area of the triangle $\Delta z_1 z_2 z_3$ is equal to 18, what is $|z_3 - z_1|$?

(A) 4 (B) 5 (C) 6 (D) 7 (E) OTA

Solution: $\arg\frac{z_2-z_1}{z_3-z_1} = \frac{\pi}{6}$ and $\left|\frac{z_2-z_1}{z_3-z_1}\right| = 2$. Since $18 = \frac{1}{2}|z_2 - z_1||z_3 - z_1|\sin\frac{\pi}{6}$, $|z_3 - z_1| = 6$.

Answer: (C)

28. Let z and w be two nonzero complex numbers satisfying $z^6 + z^3 + 1 = 0$ and $w^6 - w^3 + 1 = 0$. How many distinct complex numbers of zw are possible?

(A) 6 (B) 9 (C) 12 (D) 18 (E) NOTA

Solution: Since $z^6 + z^3 + 1 = \frac{(z^3-1)(z^6+z^3+1)}{(z^3-1)} = \frac{z^9-1}{z^3-1} = 0$, only the six roots out of the nine roots of $z^9 - 1 = 0$, which do not satisfy $z^3 - 1 = 0$, are the roots of $z^6 + z^3 + 1 = 0$. The possible z is in the form of $z = \cos\alpha + i\sin\alpha$ where $\alpha = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$. Similarly, there are six roots of $w^6 - w^3 + 1 = \frac{(w^3+1)(w^6-w^3+1)}{(w^3+1)} = \frac{w^9+1}{w^3+1} = 0$ with the form of $w = \cos\beta + i\sin\beta$ where $\beta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$. Therefore, $zw = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$ where $\alpha + \beta = \frac{\pi}{9}, \frac{3\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \cdots, \frac{17\pi}{9}$.

Answer: (B)

29. Let z_1 be the root of $z^5 = 1$ with the smallest positive imaginary part. Let z_2 be the root of $z^7 = 1$ with the smallest positive imaginary part. What is the argument of $z_1 z_2$?

(A) $\frac{2\pi}{35}$ (B) $\frac{12\pi}{35}$ (C) $\frac{24\pi}{35}$ (D) $\frac{58\pi}{35}$ (E) NOTA

Solution: $\arg(z_1) = \frac{4\pi}{5}$ and $\arg(z_2) = \frac{6\pi}{7}$, so $\arg(z_1 z_2) = \frac{58\pi}{35}$.

Answer: (D)

30. Let x and y be two nonzero complex numbers satisfying $x^2 + xy + y^2 = 0$. What is the value of $\left(\frac{x}{x+y}\right)^{100} + \left(\frac{y}{x+y}\right)^{100}$?

(A) 0 (B) -1 (C) 1 (D) 2 (E) NOTA

Solution: $x^2 + xy + y^2 = 0$ yields $\left(\frac{x}{y}\right)^2 + \frac{x}{y} + 1 = 0$. Let $w = \frac{x}{y}$, then $w^2 + w + 1 = 0$, and hence $w^3 = 1$. Now $\left(\frac{x}{x+y}\right)^{100} + \left(\frac{y}{x+y}\right)^{100} = \frac{x^{100}+y^{100}}{(x+y)^{100}} = \frac{w^{100}+1}{(w+1)^{100}} = \frac{w+1}{(-w^2)^{100}} = \frac{-w^2}{w^{200}} = \frac{-w^2}{w^2} = -1$.

Answer (B)