For all questions, answer choice E) NOTA means that none of the above answers is correct.

1. C) Since \( f(x, y) = x^2 + y^2, \ y(0) = -1 \) and \( h = \frac{1}{2} \). Use Euler’s method, we have

\[
y(h) = y(0) + hf(0, y(0)) = -1 + \frac{0^2 + (-1)^2}{2} = -\frac{1}{2}
\]

\[
y(2h) = y(h) + hf(h, y(h)) = -\frac{1}{2} + \frac{1^2 + (-\frac{1}{2})^2}{2} = -\frac{1}{4}
\]

\[
y(3h) = y(2h) + hf(2h, y(2h)) = -\frac{1}{4} + \frac{1^2 + (-\frac{1}{4})^2}{2} = \frac{9}{32}
\]

The correct answer is C.

2. C) Since \( f(x, y) = y^2 + 2x, \ y(0) = -1 \) and \( h = \frac{1}{2} \). Use Euler’s method, The end of this Euler strut

is \( (h, y(h)) = (h, y(0) + hf(0, y(0))) = (\frac{1}{2}, -\frac{1}{2}) \) The first slope is the average of

\[
\frac{f(0, y(0)) + f(h, y(h))}{2} = \frac{f(0, -1) + f\left(\frac{1}{2}, -\frac{1}{2}\right)}{2} = 1 + \frac{5}{4} = \frac{9}{8}.
\]

The correct answer is C.

3. C) It is easy to see \( y' = \frac{x^2}{1 - 3y^2} \Rightarrow (1 - 3y^2)y' = x^2 \Rightarrow 3(y - y^3) = x^3 + c \). Condition \( y(\sqrt{3}) = -2 \)

implies \( c = 15 \) so the solution is \( 3(y - y^3) = x^3 + 15 \). \( y(-\sqrt{15}) = 0 = 3y(1 - y^2) \Rightarrow y = 0, -1, 1 \).

The correct answer is C.

4. D) Let \( x = t^2, \ y = u^2, \) then \( y' = \frac{\sin \sqrt{x}}{\sqrt{y}} \) can be rewritten as \( \frac{uu'}{t} = \frac{\sin t}{u} \). The general solution is

\[
\frac{u^3}{3} = \sin t - t \cos t + c \quad \text{which means} \quad \frac{\sqrt{2}}{3} = \sin \sqrt{x} - \sqrt{x} \cos \sqrt{x} + c \quad \text{Condition} \quad y(0) = 3^2 \quad \text{implies} \quad c = 0,
\]

so the solution is \( \frac{\sqrt{2}}{3} = \sin \sqrt{x} - \sqrt{x} \cos \sqrt{x} + 1 \). \( y(\frac{\pi^2}{4}) = 6^2 \). The correct answer is D.

5. E)
6. **A)** The integrating factor of given equation is \( e^{\cos x} \). Multiply \( e^{\cos x} \) and integrating the given equation, we have the general solution is \( ye^{\cos x} = c - 2e^{\cos x} \). Condition \( y(\frac{\pi}{2}) = 1 \) implies \( c = 3 \) so the solution is \( (y + 2)e^{\cos x} = 3 \). It is easy to see \( (y + 2)e^{\cos x} = 3 \) passes through point \( (-\frac{\pi}{2}, 1) \). The correct answer is A.

7. **D)** The integrating factor is \( e^{\int 2dx} = e^{-2x} \). The correct answer is D.

8. **A)** The integrating factor is \( e^{\int \frac{1}{y}dy} = y \). The general solution is \( xy = c + y^2 \). Condition \( (0, -1) \) implies \( c = -1 \) so the solution is \( xy = 1 - y^2 \). It is easy to see \( x(1) = 0 \). The correct answer is A.

9. **C)** The integrating factor is \( e^{\int \frac{2}{xdx}} = x^2 \). The general solution is \( xy^2 = 10\int_0^x \frac{\sin t}{t} dt + c \). Using condition \( (1, 0) \) implies \( c = -10\int_0^1 \frac{\sin t}{t} dt = -10\text{Si}(1) \) so the solution is \( xy^2 = 10(\text{Si}(10) - \text{Si}(1)) \). The correct answer is C.

10. **C)** The general solution is \( 2\sin xy = (x + 1)^2 + c \). Using condition \( (0, \frac{\pi}{2}) \) implies \( c = -1 \) so the solution is \( 2\sin xy = (x + 1)^2 - 1 \). It is easy to see \( y(0) = 1, y(-1) = \frac{\pi}{6} \). The correct answer is C.

11. **A)** Since \( xy' + y = xy^3 \Rightarrow \frac{y'}{y^3} + \frac{1}{xy^2} = 1 \Rightarrow -2\frac{y'}{y^3} - 2\frac{1}{y^2} - 2\frac{1}{y^3} = \frac{-2}{x^2} \Rightarrow \frac{1}{y^2} = \frac{2}{x} + c \) Using condition \( (1, 1) \) implies \( c = -1 \) so the solution is \( y^2x(2-x) = 1 \). The correct answer is A.

12. **D)** Rewrite the given equation in the standard form \( y' = \frac{ay - 3x + \frac{y^2}{x}}{x - y} \). So the function \( f \) in this case is given by \( f(x, y) = \frac{ay - 3x + \frac{y^2}{x}}{x - y} \). The given equation is Euler homogeneous if and only if \( f(x, y) \) is scale invariant. It is easy to see \( f(cx, cy) = \frac{c(ay - 3x + \frac{y^2}{x})}{c(x - y)} = f(x, y) \) for any real number \( a \). The correct answer is D.

13. **B)** Rewrite the given equation in the form \( \frac{(y + 1)'}{(y + 1)^2} - \frac{1}{x(y + 1)} = \frac{1}{x^2} \). The general solution is \( -\frac{x}{(y + 1)} = \ln|x| + c \). Condition \( (-1, 0) \) implies \( c = 1 \) so \( -\frac{x}{(y + 1)} = \ln|x| + 1, y(1) = -2 \). The correct answer is B.
14. B) It is easy to see \( N(x, y) = x^2 + 2y, M(x, y) = axy + b \sin x \). The given equation is an exact if and only if \( \frac{\partial M}{\partial y} = ax = \frac{\partial N}{\partial x} = 2x \). Therefore, constant \( a = 2 \). B is correct answer.

15. B) It is easy to see \( N(x, y) = x^2e^y + \cos x - 1, M(x, y) = x^2e^y - y \sin x \). The equation is exact since \( \frac{\partial M}{\partial y} = 2xe^y - \sin x = \frac{\partial N}{\partial x} \). \( x^2e^y + y \cos x - y = c \) is the general. B is correct answer.

16. B) It is easy to see \( N(x, y) = x^2 + xy, M(x, y) = 3xy + y^2 \) and \( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{3x + 2y - 2x - y}{x(x + y)} = \frac{1}{x} \) which is \( y \) independent. The integrating factor is \( x \). B is correct answer.

17. C) Since \( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{4x - x}{xy} = \frac{3}{y} \) which is \( x \) independent. The integrating factor can be \( y^3 \). After we multiply the given equation by \( y^3 \), the resulting equation is \( xy^4dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0 \). The general solution is \( x^2y^4 + y^6 - 10y^4 = c \). C is correct answer.

18. D) \( x^2y^* - 3xy' + 4y = 0 \) can be rewritten in standard form \( y'' - \frac{3}{x^2} y' + \frac{4}{x^2} y = 0 \). Use 

\[
y_2 = y_1 \int e^{\int \frac{-p}{y_1^2} dx}, \text{ we have } y_2 = y_1 \int e^{\int \frac{-p}{y_1^2} dx} = x^2 \int e^{\int \frac{-p}{x^4} dx} = x^2 \ln x \]. D is correct answer.

19. E) The auxiliary equation of given equation is \( r^2 + 4r - 5 = 0 \) has roots \( r = 1 \) or \( r = -5 \). The general solution is \( y = Ae^x + Be^{-5x} \). Initial conditions implies \( A = \frac{11}{3}, B = \frac{1}{3} \), we have 

\[
y = \frac{11}{3} e^x + \frac{1}{3} e^{-5x} \]. No correct answer was provided so the correct answer is E.

20. D) The auxiliary equation of given equation is \( r^2 + 4 = 0 \) has roots \( r = 2i \) or \( r = -2i \). The general solution is \( y = A \cos 2x + B \sin 2x \). Initial condition only implies \( B = 2 \), we have 

\[
y = A \cos 2x + 2 \sin 2x \ y'(0) = 4 \]. D is correct answer.

21. C) It is easy to see \( y = \pm 1 \) satisfies equation and \( y'^2 + y^2 - 1 = 0 \Rightarrow y' = \pm \sqrt{1 - y^2} \). Integrating 

\[
y' = \sqrt{1 - y^2} \], we have \( \sin^{-1} y = x + c \). Integrating \( y' = -\sqrt{1 - y^2} \), we have \( \cos^{-1} y = x + c \). D is correct answer.
22. **C)** The critical point is \( f(x, y) = (y+1)(y-2)(4-y) = 0 \Rightarrow y = -1, y = 2, y = 4 \). C is correct answer.

23. **D)** From \( y'' = x + y - y^2 \), \( y(0) = -1 \), \( y'(0) = 1 \), we have \( y''(0) = -2 \) and \( y''' = 1 + y' - 2yy' \Rightarrow y'''(0) = 1 + 1 - 2(-1)1 = 4 \). D is correct answer.

24. **A)** From \( y'y'' = x \), \( y(0) = 1 \), \( y'(1) = \sqrt{2} \), we have \( y'^2 = x^2 + c \) by integrating. Substituting \( y'(1) = \sqrt{2} \) into general solution implies \( c = 1 \). That is \( y'^2 = x^2 + 1 \). On the other hand, we have \( y'' = \frac{x}{y'} \Rightarrow y''' = \frac{y'y'' - y'y''}{y'^2} \). Therefore, \( y'''(0) = \frac{1}{y'(0)} = 1 \). A is correct answer.

25. **D)** From \( y'' = 1 - y'^2 \), we have \( y''' = -2yy'' = 2y'(y'^2 - 1) \Rightarrow y'^4 = -2(1 - y'^2)^2 + 4y'^2y'' = 2(1 - y'^2)(3y'^2 - 1) \) Therefore, \( y'^4(0) = 2(1 - 2)(3\cdot2 - 1) = -10 \). D is correct answer.

26. **B)** From given equation, we have \( y' = 1 + 2x + 3x^2 + 4x^3 + \cdots \).

\[
y^2 = 1 + x + x^2 + x^3 + \cdots \\
+ x + x^2 + x^3 + \cdots \\
+ x^2 + x^3 + \cdots \\
+ x^3 + \cdots \\
\vdots
\]

Therefore, \( y' = y^2 \), \( y'' = 2yy' = 2y^3 \), \( y''' = 6y^2y' = 6y^4 \), \( \cdots y^{(n)} = n!y^{n+1} \). B is correct answer.

27. **B)** The auxiliary equation of given equation is \( r^2 - 2r + 1 = 0 \) has roots \( r_1 = r_2 = 1 \). We assume the form of particular solution is \( y_p = Ax^2e^x \). Substituting into given equation, we have \( y_p = \frac{x^2}{2}e^x \). B is correct answer.

28. **E)** From the given equation \( y = (x+1)y' + y'^2 \), we have \( 0 = (x+1)y'' + 2y'y'' \), which implies \( y' = c \) and \( x+1 + 2y' = 0 \). The solution is \( y = (x+1)c + c^2 \) or \( y = \frac{(x+1)^2}{4} + A \). Condition \( y(1) = -1 \) implies \( A = 0, c = -1 \). The solution is \( y = -x \) or \( y = \frac{(x+1)^2}{4} \). \( y = -x \) passes points \((0, 0)\) and \((2, -2)\) while \( y = \frac{(x+1)^2}{4} \) passes points \((-1, 0)\). E is correct answer.
29. **A)** The auxiliary equation is $r^3 + r^2 = 0$ has roots $r_1 = r_2 = 0, r_3 = -1$. We assume the given equation has a particular solution in the form $y_p = (A\sin x + B\cos x)e^x$. Substituting into the given equation, we have $y_p = \left(\frac{2}{5}\sin x + \frac{3}{10}\cos x\right)e^x$. A is correct answer.

30. **D)** From given equation $y' + y\cot x = 2\cos x, y\left(\frac{\pi}{2}\right) = 5$, we have $y\sin x = \frac{c - \cos 2x}{2}$. Condition $y\left(\frac{\pi}{2}\right) = 5$ implies $y\sin x = \frac{9 - \cos 2x}{2}$, $y\left(\frac{\pi}{2}\right) = 3$ implies $y_r\sin x = \frac{5 - \cos 2x}{2}$.

$(y(x) - y_r(x)) \bigg|_{x=\frac{\pi}{6}} = 4$. The correct answer is D.