1. If $\alpha$ and $\beta$ are two roots of the equation $5x^2 - 2x - 1 = 0$, what is $\frac{1}{\alpha} + \frac{1}{\beta}$?
   A) $\frac{1}{2}$  B) $\frac{-1}{2}$  C) $-2$  D) 2  E) NOTA

   Solution: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha \beta} = \frac{2}{-5} = -2$

   Answer: C)

2. If $\tan x + \tan y = 6$ and $\cot x + \cot y = 3$, what is $\tan(x + y)$?
   A) $-6$  B) $-5$  C) 5  D) 6  E) NOTA

   Solution: $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{6}{1 - 2} = -6$

   Answer: A)

3. Suppose that $f(4 - x) = 2x^2 - x - 7$ and $f(x) = px^2 + qx + r$. What is $p + q + r$?
   A) $-6$  B) 8  C) 14  D) $-4$  E) NOTA

   Solution: $f(1) = p + q + r = f(4 - 3) = 2(3)^2 - 3 - 7 = 8$

   Answer: B)

4. If $a_n$ is a geometric sequence with $a_1 = 2$ and $a_5 = 18$, find the sum $a_1 + a_3 + a_5 + a_7$.
   A) 80  B) 72  C) 36  D) 27  E) NOTA

   Solution: $a_5 = 2r^4 = 18$, so $r^2 = 3$.
   
   $a_1 + a_3 + a_5 + a_7 = a_1(1 + r^2 + r^4 + r^6) = 80$

   Answer: A)

5. Simplify the product: $\tan 10^\circ \tan 20^\circ \tan 30^\circ \cdots \tan 80^\circ$
   A) $\frac{1}{2}$  B) 1  C) $\frac{1}{3}$  D) 3  E) NOTA

   Solution: $\tan 10^\circ \tan 20^\circ \tan 30^\circ \cdots \tan 80^\circ = \frac{\sin 10^\circ \sin 20^\circ}{\cos 10^\circ \cos 20^\circ} \cdots \frac{\sin 80^\circ}{\cos 80^\circ} = \frac{\sin 10^\circ \sin 20^\circ \cdots \sin 80^\circ}{\sin 80^\circ \sin 70^\circ \cdots \sin 10^\circ} = 1$

   Answer: B)

6. Let $f(x) = \log_2(x + \sqrt{x^2 + 1})$. If $f(a) = b$, what is $f(-a)$?
A) \( a \quad B) \ a + b \quad C) \ b \quad D) \ -b \quad E) \ NOTA \)

Solution: 
\[
f(-a) = \log_2(-a + \sqrt{a^2 + 1})
\]
\[
= \log_2 \left( \frac{a^2 + 1 - a^2}{a + \sqrt{a^2 + 1}} \right) = \log_2 \left( \frac{1}{a + \sqrt{a^2 + 1}} \right) = \log_2 \left( a + \sqrt{a^2 + 1} \right)^{-1} = -b
\]

Answer: D)

7. Find the value of \( \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}} \cdots} \).
A) 2 \quad B) \frac{1 + \sqrt{5}}{2} \quad C) \frac{1 + \sqrt{5}}{4} \quad D) \sqrt{2} \quad E) \ NOTA \)

Solution: Let \( A = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} \). Then \( A = \sqrt{1 + A} \), so \( A \) is the positive root of 
\[
A^2 - A - 1 = 0.
\]

Answer: B)

8. Two numbers, \( x \) and \( y \), are selected at random from the interval \([0, 2] \). What is the probability that \( y \leq x + 1 \) ?
A) \( \frac{7}{8} \) \quad B) \( \frac{3}{4} \) \quad C) \( \frac{2}{3} \) \quad D) \( \frac{1}{2} \) \quad E) \ NOTA \)

Solution: The area of the triangle represented by \( y \geq x + 1 \) in the square by \( 0 \leq x \leq 2 \) and
\[
0 \leq y \leq 2 \text{ is } \frac{1}{2}.
\]

Answer: A)

9. How many real solutions are there to the equation \(|2x - 3| + |5 - 2x| = 2\)?
A) 1 \quad B) 2 \quad C) 3 \quad D) 4 \quad E) \ NOTA \)

Solution: All real values from \( \frac{3}{2} \leq x \leq \frac{5}{2} \) satisfy the equation.

Answer: E)

10. If \( \csc x - \cot x = 7 \), what is \( \csc x + \cot x \)?
A) 1 \quad B) 3 \quad C) \( \frac{1}{3} \) \quad D) \( \frac{1}{7} \) \quad E) \ NOTA \)

Solution: Since \( \csc x - \cot x = \frac{1 - \cos x}{\sin x} = 7 \).
\[
\csc x + \cot x = \frac{1 + \cos x}{\sin x} = \frac{1 - \cos^2 x}{\sin x (1 - \cos x)} = \frac{\sin x}{1 - \cos x} = \frac{1}{7}
\]

Answer: D)
11. How many prime factors are there in 999,999?
   A) 3  B) 4  C) 5  D) 6  E) NOTA

Solution: 999,999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37
Answer: C)

12. If 2 + 3i and 1 + 4i are two roots of the equation \( x^4 + ax^3 + bx^2 + cx + d = 0 \) where \( a, b, c, d \) are integers, what is the value of \( a + b + c + d \)?
   A) 145  B) 159  C) 221  D) 230  E) NOTA

Solution: The other roots are \( 2 - 3i \) and \( 1 - 4i \), so \( f(x) = x^4 + ax^3 + bx^2 + cx + d = (x^2 - 4x + 13)(x^2 - 2x + 17) \). Then \( a + b + c + d = f(1) - 1 = 159. \)
Answer: B)

13. Consider a rational function \( f(x) = \frac{ax+b}{cx+d} \) where \( c > 0 \). If \( f \) has its inverse function \( f^{-1}(x) = \frac{x+4}{2x+1} \). What is \( a + b + c + d \)?
   A) 5  B) -5  C) 4  D) -4  E) NOTA

Solution: The inverse function of \( f^{-1}(x) \) is \( f(x) \), so \( f(x) = (f^{-1})^{-1}(x) = \frac{-x+4}{2x-1} \).
Answer: C)

14. How many solutions to the equation \( \sin^4 x + \cos^4 x = 1 \) are there in the interval \([0, 2\pi)\)?
   A) 4  B) 5  C) 6  D) 7  E) NOTA

Solution: Let \( t = \sin x \), then \( t^4 + (1 - t^2)^2 = 1 \), and hence \( t = 0 \), or \( t = \pm 1 \). There are four roots of the equation over the given interval; \( x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \).
Answer: A)

15. Suppose that two positive numbers \( x \) and \( y \) satisfy \( \log_y x + \log_x y = \frac{10}{3} \) and \( xy = 81 \). What is the value of \( x + y \)?
   A) 27  B) 30  C) 36  D) 49  E) NOTA

Solution: Note that \( \log_y x + \log_x y = \log_y x + \frac{1}{\log_y x} = \frac{10}{3} \). By multiplying by \( 3 \log_y x \), we obtain \( 3 (\log_y x)^2 - 10(\log_y x) + 3 = (3 \log_y x - 1)(\log_y x - 3) = 0. \) Thus, \( \log_y x = \frac{1}{3} \) or \( \log_y x = 3 \), which yields \( x^3 = y \) or \( x = y^3 \). Then two pairs of solutions are \( x = 3, y = 27 \) or \( x = 27, y = 3 \). Thus in either case \( x + y = 30. \)
Answer: B)
16. How many 4-digit numbers are there whose digit sum equals 10?

A) 200 B) 219 C) 220 D) 286 E) NOTA

Solution: Let \( abcd \) represent a 4-digit number. Then \( a + b + c + d = 10 \) for \( 1 \leq a \leq 9 \) and \( 0 \leq b, c, d \leq 9 \). The equation is equivalent to \( a + b + c + d = 9 \) for \( 1 \leq a \leq 8 \) and \( 0 \leq b, c, d \leq 9 \). By Balls and Urns formula, there are \( \binom{9 + 4 - 1}{4 - 1} - 1 = 219 \) such 4-digit numbers.

Answer: B)

17. Suppose that \( f(x) \) is a monic polynomial of degree 3 such that \( f(1) = 1, f(2) = 4, f(3) = 9 \). Find the value of \( f(4) \).

A) 16 B) 20 C) 22 D) 29 E) NOTA

Solution: Define a new function \( g(x) \) by \( g(x) = f(x) - x^2 \). Then \( g(1) = f(1) - 1^2 = 0, g(2) = f(2) - 2^2 = 0 \) and \( g(3) = f(3) - 3^2 = 0 \). Therefore \( g(x) \) is a monic cubic polynomial having three roots 1, 2, 3. By Factor Theorem, \( g(x) = (x - 1)(x - 2)(x - 3) \). So \( f(x) = (x - 1)(x - 2)(x - 3) + x^2 \). Thus, \( f(4) = (4 - 1)(4 - 2)(4 - 3) + 4^2 = 22 \).

Answer: C)

18. Which of the following is equal to \( 3\sqrt{9 - 4\sqrt{5}} + 3\sqrt{9 + 4\sqrt{5}} \)?

A) \( 2\sqrt{3} \) B) \( 2\sqrt{5} \) C) 4 D) 3 E) NOTA

Solution: If we let \( A = 3\sqrt{9 - 4\sqrt{5}} \) and \( B = 3\sqrt{9 + 4\sqrt{5}} \), then \( A^3 + B^3 = 18 \) and \( AB = 1 \). Then \( (A + B)^3 - 3AB(A + B) = 18 \). The equation can be written as \( X^3 - 3x - 18 = 0 \), where \( X = 3\sqrt{9 - 4\sqrt{5}} + 3\sqrt{9 + 4\sqrt{5}} \). Factoring the polynomial, \( X^3 - 3x - 18 = (X - 3)(X^2 + 3X + 6) = 0 \), we obtain the value of \( X = 3 \).

Answer: D)

19. Find the sum of the solutions to the equation \( 2\sin^2 x + 5 \cdot 2\cos^2 x = 7 \) where \( x \) is in the interval \((0, 2\pi)\).

A) \( \frac{\pi}{2} \) B) \( \pi \) C) \( \frac{3\pi}{2} \) D) \( 2\pi \) E) NOTA

Solution: If we let \( X = 2\sin^2 x \), then the equation becomes \( X + 5 \cdot \frac{2}{X} = 7 \). Solving the equation for \( X \), we have \( X = 2 \) or \( X = 5 \). Since \( 0 \leq \sin^2 x \leq 1, X \leq 2 \). Thus, \( X = 2 \) and equivalently, \( \sin^2 x = 1 \). Solutions are \( \frac{\pi}{2} \) and \( \frac{3\pi}{2} \).
20. What is the remainder when $1! + 2! + 3! + \cdots + 2018!$ is divided by 7?

A) 4  B) 5  C) 6  D) 0  E) NOTA

Solution: Note $n!$ is divisible by 7 for $n \geq 7$. So the remainder is equal to the remainder in the division $1! + 2! + \cdots + 6! \equiv 1 + 2 + (-1) + 3 + 1 + (-1) \equiv 5 \pmod{7}$.

Answer: B)

21. If $a_n$ is an increasing arithmetic sequence satisfying $a_5 + a_9 = 0$ and $|a_6| = |a_7| + 2$, what is $a_1$?

A) $-12$  B) $-10$  C) $-8$  D) $-6$  E) NOTA

Solution: Since $a_n$ is an arithmetic sequence, $2a_7 = a_5 + a_9 = 0$, $a_7 = 0$. $|a_6| = |a_7| + 2 = 0 + 2 = 2$. Since $a_6 < a_7$, $a_6 = -2$. Hence $a_n = -12 + (n-1)(-2)$.

Answer: A)

22. If $x$ is a positive real number such that $\sin(\arctan\left(\frac{x}{2}\right)) = \frac{x}{3}$, what is the value of $x$?

A) 1  B) 2  C) $\sqrt{5}$  D) $\sqrt{6}$  E) NOTA

Solution: If we let $\theta = \arctan\left(\frac{x}{2}\right)$, then $\tan \theta = \frac{x}{2}$ where $\theta$ is an angle in the first quadrant. Then $\sin \theta = \sin\left(\arctan\left(\frac{x}{2}\right)\right) = \frac{x}{\sqrt{x^2+4}}$. From the given condition we obtain $\frac{x}{3} = \frac{x}{\sqrt{x^2+4}}$. Solving it for $x$, we have $x = \sqrt{5}$.

Answer: C)

23. Let $a_n$ be a sequence of integers. Suppose that $a_1, a_2, a_3$ form an arithmetic sequence and $a_2, a_3, a_4$ form a geometric sequence with an integer common ratio. If $a_4 - a_1 = 30$, what is $a_1 + a_2 + a_3 + a_4$?

A) 24  B) 33  C) 36  D) 46  E) NOTA

Solution: Let $a_2 = a$ and $a_3 = ar$ where $r$ is a common ratio. Then we can write $a_1 = 2a - ar$ and $a_4 = ar^2$. Then $a_4 - a_1 = ar^2 - (2a - ar) = a(r^2 + r - 2) = a(r + 2)(r - 1) = 30$. Since both $r + 2$ and $r - 1$ are divisors of 30 and differ by 3, $r - 1 = 2$ and $r + 2 = 5$. So $r = 3$ and $a = 3$. Thus the four terms are $-3, 3, 9, 27$. So their sum is $-3 + 3 + 9 + 27 = 36$.

Answer: C)

24. Assume that the system of equations $\begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} x \\ y \end{bmatrix}$ has a solution $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x^2 + y^2 = 1$.

What is the sum of all possible values of $k$?

A) 1  B) 3  C) 5  D) 10  E) NOTA
Solution: In order for the system to have nontrivial solutions, the matrix \[
\begin{bmatrix}
2 - k & 6 \\
2 & 1 - k
\end{bmatrix}
\] has to be nonsingular. In other words, 
\[
\det\left(\begin{bmatrix}
2 - k & 6 \\
2 & 1 - k
\end{bmatrix}\right) = \left|\begin{bmatrix}
2 - k & 6 \\
2 & 1 - k
\end{bmatrix}\right| = (2 - k)(1 - k) - 12 = 0.
\]
Solving the equation for \(k\), we have \(k = 5, -2\).

Answer: B)

25. Find the remainder when \(3^{21} + 7^{21}\) is divided by 25.

A) 0  B) 3  C) 8  D) 10  E) NOTA

Solution: If \(x = 5\), then \(3^{21} + 7^{21} = (x - 2)^{21} + (x + 2)^{21}\).
By Binomial Theorem, \((x - 2)^{21} + (x + 2)^{21} = 2x^{21} + 2 \binom{21}{19} x^{19} + \cdots + 2 \binom{21}{3} x^3 + 2 \binom{21}{1} x\). From this we know that the remainder is equal to the remainder dividing \(2 \binom{21}{1} x = 2 \binom{21}{1} 5 = 210\) by 25, which is equal to 10.

Answer: D)

26. If \(z = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\), what is the value of \((1 - z)(1 - z^2)(1 - z^3)(1 - z^4)\)?

A) 2  B) 3  C) 4  D) 5  E) NOTA

Solution: Note that \(z, z^2, z^3, z^4\) are four roots of \(x^4 + x^3 + x^2 + x + 1 = 0\). By Factor Theorem, \(x^4 + x^3 + x^2 + x + 1 = (x - z)(x - z^2)(x - z^3)(x - z^4)\). Substituting \(x = 1\), we obtain \((1 - z)(1 - z^2)(1 - z^3)(1 - z^4) = 5\).

Answer: D)

27. When \(\sqrt{15 \cdot 17 \cdot 19 \cdot 21 + 16}\) is simplified, it is a three-digit integer. What is the sum of the digits?

A) 9  B) 12  C) 15  D) 18  E) NOTA

Solution: If \(x = 18\), then \(\sqrt{15 \cdot 17 \cdot 19 \cdot 21 + 16} = \sqrt{(x - 3) \cdot (x - 1) \cdot (x + 1) \cdot (x + 3) + 16} = \sqrt{(x^2 - 9)(x^2 - 1) + 16} = \sqrt{x^4 - 10x^2 + 25} = x^2 - 5 = 18^2 - 5 = 319\)

Answer: E)

28. What is the largest integer less than or equal to the sum \(\sum_{n=1}^{2018} \log_2 \left(1 + \frac{1}{n}\right)\)?

A) 10  B) 11  C) 12  D) 13  E) NOTA
Solution: \[
\log_2 \left(1 + \frac{1}{1}\right) + \log_2 \left(1 + \frac{1}{2}\right) + \log_2 \left(1 + \frac{1}{3}\right) + \cdots + \log_2 \left(1 + \frac{1}{2018}\right) = \\
\log_2 2 + \log_2 \frac{3}{2} + \log_2 \frac{4}{3} + \cdots + \log_2 \frac{2019}{2018} = \\
\log_2 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{2019}{2018} = \log_2 2019 = 10. \cdots .
\]
Answer: A)

29. If \( a, b, c \) are positive real numbers such that \( a^3 + b^3 + c^3 = 3abc \), what is the value of \( \frac{(a+b)(b+c)(c+a)}{abc} \)?

A) 8 \hspace{1cm} B) 4 \hspace{1cm} C) 2 \hspace{1cm} D) 1 \hspace{1cm} E) NOTA

Solution: Note that \( a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \). So \( a^3 + b^3 + c^3 = 3abc \) if and if \( (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0 \). Since \( a, b, c \) are positive real numbers, \( a^2 + b^2 + c^2 - ab - bc - ca = 0 \) which implies \( a = b = c \).
Therefore, \( \frac{(a+b)(b+c)(c+a)}{abc} = \frac{2a^2b^2c^2}{a^2b^2c^2} = 8 \).

Answer: A)

30. If a nonzero \( 2 \times 2 \) matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) satisfies \( A^2 = A \) and \( A \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), which of following statements about \( A \) is NOT true?

A) \( ad - bc = 0 \)
B) \( a + d = 1 \)
C) \( A^{2018} = A \)
D) \( A^T = A \)
E) NOTA

Solution: If \( A^2 = A \), then \( A \) is singular, so \( \det(A) = ad - bc = 0 \). The trace of \( A \), \( tr(A) = a + d \), is equal to 1. \( A^n = A \) for each positive integer \( n \). But \( A \) is not necessarily symmetric.

Answer: D)