Trigonometry

1. Simplify the expression: \( \sin \left( \frac{\pi}{2} + x \right) + \cos(\pi + x) \)
   (A) 0  (B) 2 \cos x  (C) -2 \cos x  (D) 2 \sin x  (E) NOTA
   Solution: \( \sin \left( \frac{\pi}{2} + x \right) + \cos(\pi + x) = \cos x - \cos x = 0 \)
   Answer: (A)

2. If \( \sin 2\theta = \frac{7}{9} \) and \( 0 < \theta < \frac{\pi}{2} \), what is \( \sin \theta + \cos \theta \)?
   (A) \( \frac{4}{3} \)  (B) \( \frac{7}{6} \)  (C) \( \frac{5}{4} \)  (D) \( \frac{2}{3} \)  (E) NOTA
   Solution: Since \( \sin 2\theta = 2 \sin \theta \cos \theta = \frac{7}{9} \), we have
   \( (\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta = 1 + \frac{7}{9} = \frac{16}{9} \). Also, both \( \sin \theta \) and \( \cos \theta \) are positive, so \( \sin \theta + \cos \theta = \frac{4}{3} \).
   Answer: (A)

3. For a given angle \( \theta \), find the value of \( \cos \theta \) if \( \tan \theta = \frac{2}{\sqrt{5}} \) and \( \sin \theta < 0 \).
   (A) \( \frac{2}{3} \)  (B) \( -\frac{2}{3} \)  (C) \( \frac{\sqrt{5}}{3} \)  (D) \( -\frac{\sqrt{5}}{3} \)  (E) NOTA
   Solution: Since \( \tan \theta \) is positive and \( \sin \theta \) is negative, \( \theta \) is an angle in the third quadrant, which produces the cosine value of it negative. Therefore, \( \cos \theta = -\frac{\sqrt{5}}{3} \).
   Answer: (D)

4. Which of the following parametric equations represent the elliptic equation
   \( 25(x - 3)^2 + 4(y + 1)^2 = 100 \)?
   (A) \( x = 5 \cos \theta + 3, \ y = 2 \sin \theta - 1 \)  
   (B) \( x = 5 \sin \theta - 3, \ y = 4 \cos \theta + 1 \)  
   (C) \( x = 2 \cos \theta + 3, \ y = 5 \sin \theta - 1 \)  
   (D) \( x = 2 \cos \theta - 3, \ y = 5 \sin \theta + 1 \)  
   (E) NOTA
Solution: \( \frac{(x-3)^2}{4} + \frac{(y+1)^2}{25} = 1 \), so let \( \cos \theta = \frac{x-3}{2} \) and \( \sin \theta = \frac{y+1}{5} \).

Answer: (C)

5. Which one of the following is positive value when the point \( P(-4,5) \) is on the terminal side of angle \( \theta \) in standard position?

(A) \( \sin \theta \cos \theta \) \hspace{1cm} (B) \( \csc \theta \tan \theta \) \hspace{1cm} (C) \( \tan \theta \sin \theta \) \hspace{1cm} (D) \( \sin \theta \sec \theta \) 

(E) NOTA

Solution: Since the angle \( \theta \) is in the second quadrant, only \( \sin \theta \) and \( \csc \theta \) are positive and the others are negative. All values listed above are the products of positive and negative, so they are all negative.

Answer: (E)

6. Evaluate \( \sum_{n=1}^{180} \cos n^\circ \).

(A) 0 \hspace{1cm} (B) 1 \hspace{1cm} (C) 2 \hspace{1cm} (D) -1 \hspace{1cm} (E) NOTA

Solution: \( \cos(1^\circ) = -\cos(179^\circ), \cos(2^\circ) = -\cos(178^\circ), \ldots \), so

\[ \sum_{n=1}^{180} \cos n^\circ = \cos(180^\circ) = -1. \]

Answer: (D)

7. If \( \sin \theta \) and \( \cos \theta \) are two roots of an equation \( x^2 + ax + b = 0 \) for some angle \( \theta \), which of the following has to be always true?

(A) \( a^2 + 2b = -1 \) 
(B) \( a^2 - 2b = 1 \) 
(C) \( a^2 - 4b = 1 \) 
(D) \( a^2 + 4b = -1 \) 
(E) NOTA

Solution: By Vieta’s Formula \( \sin \theta + \cos \theta = -a \) and \( \sin \theta \cos \theta = b \), so

\[ a^2 - 2b = (\sin \theta + \cos \theta)^2 - 2 \sin \theta \cos \theta = 1 \]

Answer: (B)

8. Which one of the following is equal to

\[ \arcsin \left( \frac{1}{5} \right) + \arccos \left( \frac{1}{5} \right) + \arctan \left( \frac{1}{5} \right) + \arccot \left( \frac{1}{5} \right) \]
Solution: arcsin \( \left( \frac{1}{5} \right) \) and arccos \( \left( \frac{1}{5} \right) \) are complementary, and so are arctan \( \left( \frac{1}{5} \right) \) and arccot \( \left( \frac{1}{5} \right) \).

Answer: (C)

9. Find the sum of all roots of the equation \( \cos^2 x - \sin x = 1 \) where \( 0 < x < 2\pi \).

Solution: The equations is \( \sin^2 x + \sin x = 0 \). Either \( \sin x = 0 \) or \( \sin x = -1 \), so \( x = \pi \) or \( \frac{3\pi}{2} \).

Answer: (D)

10. Simplify \( \arccos \left( \cos \frac{5\pi}{4} \right) \).

Solution: \( \arccos \left( \cos \frac{5\pi}{4} \right) = \arccos \left( -\frac{1}{\sqrt{2}} \right) = \frac{3\pi}{4} \).

Answer: (B)

11. Which one of the following trigonometric expression is identical to \( \cos x \cdot (\sec x - \cos x) \)?

(A) \( \cos^2 x \)
(B) \( \sin^2 x \)
(C) \( \tan^2 x \)
(D) \( \sin x \cos x \)
(E) NOTA

Solution: \( \cos x \left( \sec x - \cos x \right) = \cos \left( \frac{1}{\cos x} \right) - \cos x = 1 - \cos^2 x = \sin^2 x \)

Answer: (B)

12. When \( \cos \theta = -\frac{5}{13} \), what is the value of \( \cos 2\theta \)?

(A) \( \frac{25}{169} \)
(B) \( -\frac{50}{169} \)
(C) \( \frac{144}{169} \)
(D) \( -\frac{119}{169} \)
(E) NOTA

Solution: \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 2 \left( -\frac{5}{13} \right)^2 - 1 = -\frac{119}{169} \)

Answer: (D)
13. What is the value of \( \sin \left(2 \arcsin \frac{1}{3}\right) \)?

(A) \( \frac{2}{3} \)  \quad (B) \( \frac{4\sqrt{2}}{3} \)  \quad (C) \( \frac{4\sqrt{2}}{9} \)  \quad (D) \( \frac{2}{9} \)  \quad (E) NOTA

Solution: Let \( \theta = \arcsin \frac{1}{3} \), then \( \sin \theta = \frac{1}{3} \), and hence \( \cos \theta = \frac{2\sqrt{2}}{3} \).

Now \( \sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = \frac{4\sqrt{2}}{9} \).

Answer: (C)

14. Given that \( \sin x - \sin y = \frac{4}{5} \) and \( \cos x + \cos y = \frac{3}{5} \), find \( \cos(x + y) \).

(A) \( \frac{3}{4} \)  \quad (B) \( -\frac{3}{4} \)  \quad (C) \( -\frac{1}{2} \)  \quad (D) \( \frac{1}{2} \)  \quad (E) NOTA

Solution: \( (\sin x - \sin y)^2 = 1 - 2 \sin x \sin y = \frac{16}{25} \),

and \( (\cos x + \cos y)^2 = 1 + 2 \cos x \cos y = \frac{9}{25} \).

By combining the two equations \( \cos(x + y) = \cos x \cos y - \sin x \sin y = -\frac{1}{2} \).

Answer: (C)

15. Which of the following angles should satisfy the inequality \( 2^\cos x + 2^\sin x < 2^\cos x^\sin x + 1 \)?

(A) 36°  \quad B) 110°  \quad C) 292°  \quad D) 310°  \quad E) NOTA

Solution: The inequality yields \( 2^\cos x \cdot 2^\sin x - 2^\cos x - 2^\sin x + 1 > 0 \) which is equivalent to \( (2^\cos x - 1)(2^\sin x - 1) > 0 \). Thus, either \( \cos x > 0 \) and \( \sin x > 0 \), or \( \cos x < 0 \) and \( \sin x < 0 \). Finding an angle located in the first or the third quadrant, the answer is (A).

Answer: (A)

16. Let \( f(x) = \sin x \) and \( g(x) = \cos x \) be two functions defined on \([0, \frac{\pi}{2}]\). Which of the four functions, \( f(f(x)), g(g(x)), g(f(x)), g(g(x)) \), are increasing over \([0, \frac{\pi}{2}]\)?

(A) \( f(g(x)) \) and \( g(f(x)) \)
(B) \( f(f(x)) \) and \( g(g(x)) \)
(C) \( f(g(x)) \) and \( g(g(x)) \)
(D) \( f(f(x)) \) and \( g(f(x)) \)
Solution: Note that $f(x)$ is increasing on $[0, \frac{\pi}{2}]$, and $g(x)$ is decreasing on $[0, \frac{\pi}{2}]$ where ranges of both functions lie in $[0, \frac{\pi}{2}]$. Then the composition of an increasing function with an increasing function or the composition of a decreasing function with a decreasing function yields an increasing function on the given interval.

Answer: (B)

17. If $\tan \theta + \cot \theta = 5$, what is the value of $\csc^2 \theta + \sec^2 \theta$?

(A) 2    (B) 5    (C) 23    (D) 25    (E) NOTA

Solution: $\tan^2 \theta + 2 + \cot^2 \theta = \sec^2 \theta + \csc^2 \theta = 25$

Answer: (D)

18. When the solution set of the equation $\lfloor \sin x \rfloor + \lfloor 2\sin x \rfloor + \lfloor 3\sin x \rfloor = 1$ for $x$ in $[0, \frac{\pi}{2}]$ is written as $\alpha \leq x < \beta$, what is $\cos(\alpha + \beta)$?

(A)$\frac{\sqrt{3}}{3} + \frac{1}{6}$    (B)$\frac{\sqrt{3}}{3} - \frac{1}{6}$    (C)$\frac{\sqrt{6}}{6} + \frac{1}{3}$    (D)$\frac{\sqrt{6}}{6} - \frac{1}{3}$    (E) NOTA

Solution: Note that $3 \sin x \geq 1$ and $2 \sin x < 1$. So $\frac{1}{3} \leq \sin x < \frac{1}{2}$ or equivalently,

$$\arcsin \frac{1}{3} \leq x < \frac{\pi}{6}.$$  Therefore $\alpha = \arcsin \frac{1}{3}$ and $\beta = \frac{\pi}{6}$.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{2\sqrt{2}}{3} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{3} - \frac{1}{6}.$$

Answer: (B)

19. Which of the following intervals can be a domain of the function $f(x) = \frac{1}{\sqrt{1 - 4\sin^2 x}}$?

(A)$-\frac{\pi}{3} < x < \frac{\pi}{3}$    (B)$\frac{\pi}{3} < x < \frac{2\pi}{3}$    (C)$\frac{\pi}{6} < x < \frac{5\pi}{6}$    (D)$\frac{5\pi}{6} < x < \frac{7\pi}{6}$    (E) NOTA

Solution: $1 - 4 \sin^2 x > 0$, so $-\frac{1}{2} < \sin x < \frac{1}{2}$. Solving the triangular inequality, we obtain solution sets like $-\frac{\pi}{6} < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{7\pi}{6}$.

Answer: (D)
20. Which pair of the following graphs coincide?
   a) \( y = 3 \sin 2 \left( x - \frac{\pi}{4} \right) \)
   b) \( y = -3 \sin 2x \)
   c) \( y = -3 \cos 2x \)
   d) \( y = 3 \cos 2 \left( x - \frac{\pi}{4} \right) \)

   (A) a and b  (B) b and c  (C) c and d  (D) a and c  (E) NOTA

   Solution: \( 3 \sin 2 \left( x - \frac{\pi}{4} \right) = 3 \sin \left( 2x - \frac{\pi}{2} \right) = 3 \sin(2x) \cos\left( \frac{\pi}{2} \right) - 3 \cos(2x) \sin\left( \frac{\pi}{2} \right) = -3 \cos(2x) \)

   Answer: (D)

21. Four points \( A, B, C, D \) lie on the circumference of a circle to form a quadrilateral. Let \( \alpha, \beta, \gamma, \delta \) denote four interior angles of the quadrilateral associated with \( A, B, C, D \), respectively. Which of the following is NOT true?
   (A) \( \cos \beta \cos \delta = \sin \beta \sin \delta + 1 \)
   (B) \( \sin \alpha \cos \gamma + \cos \alpha \sin \gamma = 0 \)
   (C) \( \sin^2 \alpha + \cos^2 \gamma = 1 \)
   (D) \( \cos \beta + \cos \delta = 0 \)
   (E) NOTA

   Solution: Note that \( \alpha + \gamma = \pi \) and \( \beta + \delta = \pi \). (C) and (D) follow from the fact immediately. And \( \sin \alpha \cos \gamma + \cos \alpha \sin \gamma = \sin(\alpha + \gamma) = \sin \pi = 0 \) which shows (B). However, \( \cos \beta \cos \delta - \sin \beta \sin \delta = \cos(\beta + \delta) = \cos \pi = -1 \).

   Answer: (A)

22. What is \( \cot 80^\circ \cot 55^\circ + \cot 80^\circ + \cot 55^\circ \)?
   (A) 0  (B) 1  (C) 2  (D) 3  (E) NOTA

   Solution: Since \( \tan 135^\circ = \frac{\tan 55^\circ + \tan 80^\circ}{1 - \tan 55^\circ \tan 80^\circ} = -1 \),

   \[ \tan 55^\circ + \tan 80^\circ = -1 + \tan 55^\circ \tan 80^\circ. \] 

   Now,
cot 80° cot 55° + cot 80° + cot 55° = \frac{1}{\tan 80° \tan 55°} + \frac{1}{\tan 80°} + \frac{1}{\tan 55°}
= \frac{1 + \tan 80° + \tan 55°}{\tan 80° \tan 55°} = \frac{1 + (-1 + \tan 80° \tan 55°)}{\tan 80° \tan 55°} = 1

\text{Answer: (B)}

23. Let \( a_n \) be a sequence which represents the number of intersecting points of two graphs, \( y = \sin x \) and \( y = \cos 2nx \), over the open interval \((0, 2\pi)\). Write the general term of the sequence \( a_n \).

(A) 2n  \hspace{1cm} (B) 4n  \hspace{1cm} (C) 4n - 1  \hspace{1cm} (D) 8n - 5  \hspace{1cm} (E) NOTA

Solution: The period of the graph of \( y = \cos 2nx \) is \( \frac{\pi}{n} \), so there are \( 2n \) complete cycle of cosine graphs between 0 and \( 2\pi \). Two graphs of \( y = \sin x \) and \( y = \cos 2nx \) are tangent either at \( x = \frac{\pi}{2} \) when \( n \) is even or at \( x = \frac{3\pi}{2} \) when \( n \) is odd. Thus, the number of intersections is \( 2(2n) - 1 \).

Answer: (C)

24. Which of the following is equal to the infinite sum 
\( \sin x + \sin x \cos^2 x + \sin x \cos^4 x + \sin x \cos^6 x + \cdots \) for \( x \) in \((0, \pi)\) ?

(A) \( \sin x \) \hspace{1cm} (B) \( \csc x \) \hspace{1cm} (C) \( \cos x \) \hspace{1cm} (D) \( \cos x \) \hspace{1cm} (E) NOTA

Solution:
\[
\sin x + \sin x \cos^2 x + \sin x \cos^4 x + \sin x \cos^6 x + \cdots = \frac{\sin x}{1 - \cos^2 x} = \frac{1}{\sin x} = \csc x
\]

Answer: (B)

25. How many solutions to the equation \( \cos^2 x - 3 \cos x - 4 = 0 \) are there in the open interval \((0, 2\pi)\)?

(A) 1  \hspace{1cm} (B) 2  \hspace{1cm} (C) 3  \hspace{1cm} (D) 4  \hspace{1cm} (E) NOTA

Solution: Since \( \cos^2 x - 3 \cos x - 4 = (\cos x - 4)(\cos x + 1) = 0 \) and \(-1 \leq \cos x \leq 1\) for all \( x \), we have \( \cos x = -1 \), and hence there is only one root, \( x = \pi \) on the interval \((0, 2\pi)\).

Answer: (A)

26. Aaron and Bill watch a drone flying 120 feet above the ground. The angle of the elevation from Aaron to the drone is \( 45^\circ \) and from Bill to the drone is \( 60^\circ \). Assuming that the
positions of Aaron and Bill and the point of perpendicular projection from the drone to the ground form a line, how far are Aaron and Bill apart?

(A) $120 - 40\sqrt{3}$  
(B) $120 - 120\sqrt{3}$  
(C) $120 + \sqrt{3}$  
(D) $120\sqrt{3}$  
(E) NOTA

Solution: Let $x$ be the distance from Aaron to Bill and let $y$ be the distance from Bill to the projection point of the drone on the ground. Then $x + y = 120$ and $y = \frac{120}{\tan 60^\circ} = 40\sqrt{3}$, and hence $x = 120 - 40\sqrt{3}$

Answer: (A)

27. Let $F_n$ be the sequence with $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$. Define a sequence, $z_n$, of complex numbers by $z_n = \cos F_n + i \sin F_n$. Which of the following is true for $z_n$?

(A) $z_{n+2} = z_{n+1} + z_n$  
(B) $z_{n+1} = 2z_n$  
(C) $z_{n+2} = z_{n+1}z_n$  
(D) $z_n^2 = z_{2n}$  
(E) NOTA

Solution:

$$z_{n+2} = \cos F_{n+2} + i \sin F_{n+2} = \cos(F_{n+1} + F_n) + i \sin(F_{n+1} + F_n)$$

$$= (\cos F_{n+1} + i \sin F_{n+1})(\cos F_n + i \sin F_n) = z_{n+1}z_n$$

Answer: (C)

28. Let $G_n$ be the sequence with $G_1 = 1, G_{n+1} = 2G_n$. Define a sequence, $w_n$, of complex numbers by $w_n = \cos G_n + i \sin G_n$. Which of the following is true for $w_n$?

(A) $w_{n+2} = w_{n+1} + w_n$  
(B) $w_{n+1} = 2w_n$  
(C) $w_{n+2} = w_{n+1}w_n$  
(D) $w_n^2 = w_{2n}$  
(E) NOTA

Solution: $w_n^2 = (\cos G_n + i \sin G_n)^2 = \cos 2G_n + i \sin 2G_n = w_{n+1}$

Answer: (D)

29. Which of the following is equal to $\cos \frac{2\pi}{5}$?

(A) $\frac{\sqrt{5} + 1}{4}$  
(B) $\frac{\sqrt{5} - 1}{4}$  
(C) $\frac{\sqrt{6} + \sqrt{2}}{4}$  
(D) $\frac{\sqrt{6} - \sqrt{2}}{4}$  
(E) NOTA
Let \( \theta = \frac{2\pi}{5} \), then \( 5\theta = 2\pi \). Since \( \sin 3\theta = \sin(2\pi - 2\theta) = \sin 2\theta \) and \( \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \), we have \( 2 \sin \theta \cos \theta = 3 \sin \theta - 4 \sin^3 \theta \), and so \( 3 - 4 \sin^2 \theta = 2 \cos \theta \) which yields a quadratic equation \( 4 \cos^2 \theta - 2 \cos \theta - 1 = 0 \). Solving this equation for \( \cos \theta \), we obtain the positive value of \( \cos \theta = \frac{-1 + \sqrt{5}}{4} \).

Answer: (B)

30. Evaluate the product: \( \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ \)

(A) \( \frac{1}{8} \)  (B) \( \frac{1}{16} \)  (C) \( \frac{1}{32} \)  (D) \( \frac{1}{64} \)  (E) NOTA

Solution: \( \sin 10^\circ \sin 70^\circ \sin 30^\circ \sin 50^\circ \)

\[
= \frac{\sin 10^\circ \sin 20^\circ \sin 30^\circ \sin 40^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ \sin 80^\circ}{\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ}
\]

\[
= \frac{\sin 10^\circ \sin 20^\circ \sin 30^\circ \sin 40^\circ \cos 40^\circ \cos 30^\circ \cos 20^\circ \cos 10^\circ}{2 \sin 10^\circ \cos 10^\circ \cdot 2 \sin 20^\circ \cos 20^\circ \cdot 2 \sin 30^\circ \cos 30^\circ \cdot 2 \sin 40^\circ \cos 40^\circ}
\]

\[
= \frac{1}{16}
\]

Answer: (B)