NOTA means none of the above answers is correct. Problems will specify the units of answer choices if necessary. $x \in [a, b]$ means $a \leq x \leq b$. Good luck!

Use the following information for questions 1-5:
A rocket is launched vertically from rest. Assume that immediately following its launch, the rocket has some vertical velocity v_y, and no horizontal velocity $v_x = 0$.

1. The launch pad is 200m horizontal from your position. If the angle of elevation of your line of sight to the rocket is increasing at a rate of 1rad/s, what is the vertical velocity of the rocket in m/s when the angle of elevation is $\pi/4$?

A. 400 B. 200 C. 100 D. 50 E. NOTA

2. After one second (starting at $t = 1$), a booster provides a horizontal acceleration for two seconds given by $a(t) = 2/t$. What is the final horizontal velocity?

A. $2\ln 2$ B. $2(\ln 2 + 1)$ C. $2\ln 3$ D. $2/3$ E. NOTA

3. Approximate the horizontal distance traveled from $t = 1$ to $t = 3$ using trapezoidal rule with $n = 2$ and the equation for velocity derived in the previous problem.

A. $2\ln 12$ B. $\ln 12$ C. $2\ln 6$ D. $\ln 6$ E. NOTA

4. Once the rocket reaches apogee (it's highest point of flight), a parachute deploys. The parachute is a paraboloid, given by rotating the parabola $y = \frac{1}{2}x^2,\ x \in [-2, 2]$ around the y axis. What is the volume of air the parachute contains?

A. $\frac{8}{5}\pi$ B. 4π C. 8π D. 16π E. NOTA

5. The rocket begins to fall back to the ground, pulled down by a weight of $100N$. The drag force exerted upwards on the rocket is given by $f_{drag} = \frac{1}{2}\rho C A v^2$ where $\rho = \frac{1kg}{m^3}$ and $C = 1$. Assume the projected área A (in m^2) is the area of the bottom of the parachute (intersection of the plane $y = 2$). No other forces are present. What is the rocket's terminal velocity v?

A. $10\frac{m}{s}$ B. $10\sqrt{2}\frac{m}{s}$ C. $20\frac{m}{s}$ D. $200\frac{m}{s}$ E. NOTA

Use the following information for questions 6-11:
You wish to launch a satellite into low earth orbit (LEO). The mass of the rocket you are using is $500{,}000kg$ and the required mass of the payload is exactly $3000kg$. The rocket's height $h(t)$ and down range distance $x(t)$ are given by the following parametric equations:

$$h(t) = 2000 - t^3 + 100t^2 + 3t$$
$$x(t) = 100e^t$$

6. The satellite you want to launch is shaped like a regular triangular prism. Assume the density of the satellite is $1200\frac{kg}{m^3}$. Which answer is closest to the minimum surface area of the satellite you can achieve given these constraints (all answers in m^2)?

   A. 10 B. 12 C. 14 D. 15 E. NOTA

7. Which of the following is closest to the magnitude of the velocity of the rocket one time unit after it launches?

   A. 250 B. 350 C. 450 D. 550 E. NOTA

8. During which time interval does the rocket stop going up (reach apogee)?

   A. $t \in [30, 40]$ B. $t \in [60, 70]$ C. $t \in [90, 110]$ D. $t \in [190, 210]$ E. NOTA

9. Once the rocket reaches sufficient altitude, its second stage begins another burn to position the satellite. You are given

$$Ma = -u\frac{dM}{dt}$$

   where M is the mass of the rocket, a is it's acceleration, and u is the constant exhaust velocity. Which of the following best approximates the change in velocity of the rocket as a result of the burn of $85{,}000kg$ of fuel if the initial mass of fuel is $135{,}000kg$ (neglecting units)?

   A. $\ln u$ B. $u(1 - e^{-t})$ C. $\frac{1}{2}u$ D. u E. NOTA

10. The rocket fuel is in a spherical tank with radius 3. Find the equation for the volume of fuel remaining in terms of the height h from the bottom of the tank. Approximate the depth of fuel when the tank is ¼ full, using Newton's method, starting with $h_0 = 2$ and taking one step.

    A. $47/24$ B. $49/24$ C. $34/15$ D. $19/15$ E. NOTA

11. Assume fuel is being used constantly at a rate of r from the tank in the previous problem. How fast is the height decreasing when the tank is half full?

    A. $\frac{r}{3\pi}$ B. $\frac{r}{6\pi}$ C. $\frac{r}{6.75\pi}$ D. $\frac{r}{9\pi}$ E. NOTA

Use the following information for question 12:
The root mean square (RMS) voltage of some voltage $v(t)$ is calculated by:

1. Squaring the function $v(t)$
2. Calculating the mean of the function obtained in step 1
3. Taking the square root of the value obtained in step 2

12. The voltage coming out of outlets in the United States has a root mean square (RMS) value of 120V. Assuming $v(t) = A\sin(t)$, what is its amplitude? *Aside: we quote the RMS values instead of the peak values due to its usefulness in power calculations.*

A. $120\sqrt{2}$ B. 240 C. $240\sqrt{2}$ D. 240π E. NOTA

13. You have an electrical device containing a battery with constant voltage V and constant internal resistance r. You connect a resistor with resistance R, so the current flowing through this resistor is $i = \frac{V}{r+R}$, and the voltage across the resistor is iR. What is the resistance value R that maximizes the power dissipated across this resistor, given that power is the current through the resistor times the voltage across the resistor?

A. $\frac{r}{2r+1}$ B. $\frac{r}{2}$ C. r D. ∞ E. NOTA

Use the following information for question 14-15:
The probability of battery failure has probability density function $f(t) = \lambda e^{-\lambda t}$ for positive constant λ at time $t \geq 0$.

14. You empirically determine the mean lifetime of a battery to be 20 hours. What is λ? (in #/hr)

A. $1/19$ B. $1/\sqrt{19}$ C. $1/20$ D. $1/\sqrt{20}$ E. NOTA

15. What is the probability of a battery lasting more than 20 hours?

A. e^{-1} B. $1/2$ C. $1-e^{-1}$ D. $1-e^{-\sqrt{20}}$ E. NOTA

16. The area reached by a radio station is bounded by the curves $y = \sqrt{9-9x^2}$ and $y = -\sqrt{4-4x^2}$. Find the total area reached.

A. 4π B. 5π C. 6π D. 9π E. NOTA

17. Now assume a radio transmitter has a directional range given by the Cardioid with polar equation $r(\theta) = 2 + 2\cos\theta$. What is the area of this region?

A. $\frac{3\pi}{2}$ B. 3π C. 6π D. 12π E. NOTA

18. A radio transmitter is at $(-1, 2)$ behind a hill modeled in the first quadrant by $y = x(2-x)$. What is the smallest x coordinate on the positive x-axis that still has a line of sight to the transmitter?

A. $2+\sqrt{5}$ B. $-1+\sqrt{5}$ C. 3 D. $1+\sqrt{5}$ E. NOTA

19. The fraction of the city population that listens to a certain radio station during primetime is given by $P(t) = \frac{t}{(t+2)^2}$. What is the peak fraction of the population listening during primetime?

A. $3/32$ B. $1/9$ C. $1/8$ D. $2/9$ E. NOTA

Use the following information for questions 20-21:
The number of listeners for this particular station, **in thousands**, $L(t)$, grows at a rate directly proportional to $L(t)\big(780 - 12L(t)\big)$. Initially there are 1000 listeners, and after one year there are 10,000 listeners.

20. How many listeners can this radio station expect to have after a "long time" ($\lim_{t\to\infty} L(t)$)?

A. $32{,}500$ B. $65{,}000$ C. $100{,}000$ D. $780{,}000$ E. NOTA

21. How many listeners will the station have after two years (to the nearest five-thousand)?

A. $19{,}000$ B. $20{,}000$ C. $45{,}000$ D. $100{,}000$ E. NOTA

22. Consider some function for a signal $f(t)$. Let $f_e(t)$ and $f_o(t)$ be the even and odd components of $f(t)$ respectively. Given that $f_h(t) = f_e(t) - jf_o(t)$, where j is the complex unit, for all t, find the value of the energy of $f_h(t)$ **in simplest form**, given by the integral $\int_{-\infty}^{\infty} f_h^2(t)dt$. Assume all integrals converge.

A. $\int_{-\infty}^{\infty}\big(f_e^2(t) - f_o^2(t)\big)dt$
B. $\int_{-\infty}^{\infty}\big(f_e^2(t) + f_o^2(t)\big)dt$
C. $\int_{-\infty}^{\infty}\big(f_e^2(t) - 2jf_e(t)f_o(t) - f_o^2(t)\big)dt$
D. $\int_{-\infty}^{\infty}\big(f_e^2(t) - 2jf_e(t)f_o(t) + f_o^2(t)\big)dt$
E. NOTA

23. How many of the following statements are true?

I. If the sum of two functions $f_1(t) + f_2(t)$ is periodic, $f_1(t)$ and $f_2(t)$ must be periodic.
II. If $f_1(t)$ and $f_2(t)$ are periodic, their sum $f_1(t) + f_2(t)$ must be periodic.
III. $g(t) = \cos(2\pi t) + \sin(\pi t)$ is periodic.
IV. $h(t) = \cos(2\pi t)\sin(2t)$ is periodic.

A. 1 B. 2 C. 3 D. 4 E. NOTA

Use the following information for questions 24-29:
The energy of a signal $x(t)$ is given by

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

and its power is given by

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

which is the average value of the energy of a signal. An energy signal has energy $0 < E_x < \infty$, and a power signal has power $0 < P_x < \infty$. These two signals require different means of analysis.

24. Classify the signal $x(t) = \sin(t)$ in these terms.

A. Energy B. Power C. Both D. Neither E. NOTA

25. Classify the following signal in these terms:

$$x(t) = \begin{cases} 1 & t \in [2^n, 2^n + 1] \text{ for } n \in \{0, 1, 2, \dots\} \\ 0 & \text{otherwise} \end{cases}$$

(Hint: Draw a picture. This looks like a series of pulses with progressively longer delays between them.)

A. Energy B. Power C. Both D. Neither E. NOTA

26. Determine the value of the signal $x(t) = \sin(t^2)$ at $t = 2$ using a two term Taylor Polynomial for $x(t)$.

A. $2/3$ B. $20/9$ C. $20/3$ D. $52/3$ E. NOTA

Use the following information for questions 27-29

The Fourier transform of an energy signal $f(t)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

where $j = \sqrt{-1}$ and ω is real. This gives us a representation of $f(t)$ in the frequency domain. Due to conservation of energy, we must have the energy in the time domain equal to the energy in the frequency domain (the $\frac{1}{2\pi}$ scales appropriately for the conversion to radians/sec):

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

We will explore the following signal for positive, real-valued *a*:

$$x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

27. What is the energy of the signal?

A. $\frac{1}{2a}$ B. $\frac{1}{a}$ C. $\frac{2}{a}$ D. ∞ E. NOTA

28. What is the Fourier transform of the signal?

A. $\frac{a}{a-j\omega}$ B. $\frac{1}{a-j\omega}$ C. $\frac{1}{a+j\omega}$ D. $\frac{1}{j\omega-a}$ E. NOTA

29. What is bandwidth required to capture 2/3 of the signal's energy? In other words, find the W such that

$$\frac{1}{2\pi}\int_{-W}^{W} |X(\omega)|^2 d\omega = 2/3(E_x)$$

A. $a/2$ B. $a/\sqrt{2}$ C. a D. $a\sqrt{3}$ E. NOTA

30. Congratulations! You're just about done. One last question: which of the following describe the function $f(x) = -x^3 + x$ at the point $(1, 0)$?

A. Increasing, concave up
B. Decreasing, concave up
C. Increasing, concave down
D. Decreasing, concave down
E. NOTA