(1) SOLUTION: $\lim_{x\to 3}\frac{x^2-x-6}{x^2-4x+3}=\lim_{x\to 3}\frac{(x-3)(x+2)}{(x-1)(x-3)}=\lim_{x\to 3}\frac{(x+2)}{(x-1)}=\frac{5}{2}$. B.

(2) SOLUTION: From compound interest, $\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{tx}=e^{rt}$. C.

(3) SOLUTION: $y=\frac{\sin(x)}{x}\to y'=\frac{x\cdot\cos(x)-\sin(x)}{x^2}\to y'(\pi)=-\frac{1}{\pi}\to$ The tangent line is $y-0=-\frac{1}{\pi}(x-\pi)\to y=-\frac{1}{\pi}x+1$. A.

(4) SOLUTION: $y=x\cdot\ln(x^2+1)\to y'=\ln(x^2+1)+x\cdot\frac{2x}{x^2+1}\to y'(1)=\ln(2)+1\cdot\frac{2}{1+1}=\ln(2)+1$. So the normal slope is $-\frac{1}{\ln(2)+1}$. B.

(5) SOLUTION: $y^3+xy^2+x^2=13\to 3y^2y'+y^2+2xyy'+2x=0\to y'=\frac{-2x-y^2}{2xy+3y^2}\to y'$ at $(1,2)$ is $\frac{-2-2^2}{2\cdot 2+3\cdot 2^2}=-\frac{6}{16}=-\frac{3}{8}$. D.

(6) SOLUTION: Four rectangles of equal width on x = 1 to 3 means x = 1, 3/2, 2, and 5/2. So $A\approx\left(\frac{1}{2}\right)*(1^2+1+1)+\left(\frac{1}{2}\right)*\left(\left(\frac{3}{2}\right)^2+\frac{3}{2}+1\right)+\left(\frac{1}{2}\right)*(2^2+2+1)+\left(\frac{1}{2}\right)*\left(\left(\frac{5}{2}\right)^2+\frac{5}{2}+1\right)=\frac{3}{2}+\frac{19}{8}+\frac{7}{2}+\frac{39}{8}=\frac{49}{4}$. B.

(7) SOLUTION: $\int_1^3(x^3-4x+3)dx=\left[\frac{1}{4}x^4-2x^2+3x\right]_1^3=\left(\frac{81}{4}-2(9)+3(3)\right)-\left(\frac{1}{4}-2(1)+3(1)\right)=20-16+6=10$. A.

(8) SOLUTION: $\int_0^1\left(\sqrt[3]{x}-x^3\right)dx=\left[\frac{3}{4}x^{\frac{4}{3}}-\frac{1}{4}x^4\right]_0^1=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$. C.

(9) SOLUTION: $\int_0^{\sqrt{\ln(2)}}xe^{x^2}dx=\int_0^{\ln(2)}\frac{1}{2}e^u du=\frac{1}{2}[e^u]_0^{\ln(2)}=\frac{1}{2}(2-1)=\frac{1}{2}$. C.

(10) SOLUTION: $\frac{d}{dx}\int_{\sqrt{\pi x^4}}^{\sqrt{\arctan(x)}}\cos(t^2)\,dt=\cos(\arctan(x))*\left(\frac{\frac{1}{1+x^2}}{2\sqrt{\arctan(x)}}\right)-4\sqrt{\pi}x^3\cos(\pi x^8)$. So $f'(1)=\cos(arctan(1))*\left(\frac{\frac{1}{2}}{2\sqrt{\arctan(1)}}\right)-4\sqrt{\pi}\cos(\pi)=\cos\left(\frac{\pi}{4}\right)*\left(\frac{\frac{1}{2}}{2\sqrt{\frac{\pi}{4}}}\right)+4\sqrt{\pi}=\frac{\sqrt{2}}{4\sqrt{\pi}}+4\sqrt{\pi}=\frac{\sqrt{2}+16\pi}{4\sqrt{\pi}}$. B.

(11) SOLUTION: The area of the rectangle is $A(x)=\frac{2x}{1+x^2}$. So $A'(x)=\frac{2(1+x^2)-2x(2x)}{(1+x^2)^2}=0\to 1-x^2=0\to x=1\to A=1$. A.

(12) SOLUTION: Average value is $\frac{1}{\frac{\pi}{4}-\frac{-\pi}{4}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec(x)\,dx=\frac{2}{\pi}*2*[\ln|\sec(x)+\tan(x)|]_0^{\frac{\pi}{4}}=\frac{4}{\pi}\left(\ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right|\right)-\frac{4}{\pi}(\ln|\sec(0)+\tan(0)|)=\frac{4}{\pi}\ln\left(\sqrt{2}+1\right)$. D.

(13) SOLUTION: Let $u = \sqrt{2+\sqrt{x}} \rightarrow (u^2-2)^2 = x \rightarrow 2(u^2-2) * 2udu = dx$. So $\int_4^{49} \sqrt{2+\sqrt{x}}dx = \int_2^3 u * 2(u^2-2) * 2udu = 4\int_2^3 (u^4 - 2u^2)du = 4\left[\frac{1}{5}u^5 - \frac{2}{3}u^3\right]_2^3 = 4\left(\frac{243}{5} - 18 - \frac{32}{5} + \frac{16}{3}\right) = 4\left(\frac{443}{15}\right) = \frac{1772}{15}$. C.

(14) SOLUTION: To use differentials we use the fact that $f(x+\Delta x) \approx f(x) + \Delta x f'(x)$. So, $\sqrt[10]{e} = e^{0+0.1} = e^0 + e^0 * 0.1 = 1.1$. C.

(15) SOLUTION: The integrand has an asymptote at x=2, causing this integral not to converge. E.

(16) SOLUTION: The area of a semi-circle of diameter d is $A = \frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{8}d^2$. So $V = \int_0^\pi \frac{\pi}{8}\sin^2(x)\,dx = \frac{\pi}{8}\int_0^\pi \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx = \frac{\pi}{8}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_0^\pi = \frac{\pi^2}{16}$. A.

(17) SOLUTION: Using Shell Method, we have $V = 2\pi\int_0^\infty xe^{-x}dx$. Using integration by parts with $u = x \rightarrow du = dx$ and $dv = e^{-x}dx \rightarrow v = -e^{-x}$, we get $2\pi\int_0^\infty xe^{-x}dx = 2\pi[-xe^{-x}]_0^\infty - 2\pi\int_0^\infty -e^{-x}dx = 2\pi[-e^{-x}]_0^\infty = 2\pi$. D.

(18) SOLUTION: A point on the edge of the wheel corresponds to $(\cos(\theta), \sin(\theta))$, so the distance is $D = \sqrt{(\cos(\theta)-3)^2 + (\sin(\theta)-3)^2} \rightarrow \frac{dD}{dt} = \frac{(\cos(\theta)-3)*(-\sin(\theta)) + (\sin(\theta)-3)*(\cos(\theta))}{\sqrt{(\cos(\theta)-3)^2 + (\sin(\theta)-3)^2}} * \frac{d\theta}{dt} = 4\frac{3\sin(\theta) - 3\cos(\theta)}{\sqrt{(\cos(\theta)-3)^2 + (\sin(\theta)-3)^2}} = 4\frac{-3}{\sqrt{(1-3)^2 + (-3)^2}} = \frac{-12}{\sqrt{4+9}} = \frac{-12\sqrt{13}}{13}$. B.

(19) SOLUTION: $f(x) = x + \frac{d}{dx}\big(f(x)\big)$ so $f(x) = x + f'(x) \rightarrow f'(x) = f(x) - x \rightarrow f''(x) = f'(x) - 1 = f(x) - x - 1 \rightarrow f''(1) = f(1) - 1 - 1 = 3$. A.

(20) SOLUTION: $\int_{-1}^0 \frac{x^2+4x+5}{x^2+2x+2}dx = \int_{-1}^0 \frac{x^2+2x+2}{x^2+2x+2} + \frac{2x+2}{x^2+2x+2} + \frac{1}{x^2+2x+2}dx = \int_{-1}^0 1 + \frac{2x+2}{x^2+2x+2} + \frac{1}{(x+1)^2+1}dx = [x + \arctan(x+1)]_{-1}^0 + [\ln(u)]_1^2 = 0 + \frac{\pi}{4} - -1 - 0 + \ln(2) - \ln(1) = 1 + \frac{\pi}{4} + \ln(2)$. C.

(21) SOLUTION: $\int_0^{-5} f(x)dx = 7 \rightarrow \int_{-5}^0 f(x)dx = -7 \rightarrow \int_0^5 f(x)dx = 7 \rightarrow \int_{-4}^5 f(x)dx = 3 = \int_{-4}^0 f(x)dx + \int_0^5 f(x)dx = \int_{-4}^0 f(x)dx + 7 \rightarrow \int_{-4}^0 f(x)dx = -4 \rightarrow \int_0^4 f(x)dx = 4$. B.

(22) SOLUTION: $\frac{d}{du}(f^{-1}(u)) = \frac{1}{f'(f^{-1}(u))}$. $f(x) = 8$ when $x = 1$ by inspection, and f is only increasing due to always positive first derivative, so $f^{-1}(8) = 1$. $f'^{(x)} = 3x^2 + 4x^2 + 3$, so $f'(1) = 3 + 4 + 3 = 10$. So the answer is $\frac{1}{10}$. C.

(23) SOLUTION: I is false, because the nth term test can only show something diverges, not that it converges. II is true because $\lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1 > 0$ and finite. III is true because

$\int_1^\infty \frac{x}{x^2+1}dx = \left[\frac{1}{2}\ln(x^2+1)\right]_1^\infty = \infty$. IV is false because $\lim_{n\to\infty} \frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \lim_{n\to\infty} \frac{(n+1)(n^2+1)}{n(n^2+2n+2)} = 1$, so no conclusion can be reached. V is false because if $\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right)^{\frac{1}{n}} = L$ then $\ln(L) = \lim_{n\to\infty}\left(\frac{1}{n}\ln\left(\frac{n}{n^2+1}\right)\right) = \lim_{n\to\infty}\left(\frac{\ln(n)-\ln(n^2+1)}{n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n}-\frac{2n}{n^2+1}}{1}\right) = \lim_{n\to\infty}\left(\frac{1-n^2}{n(n^2+1)}\right) = 0$, so $L=1$ and no conclusion can be reached. D.

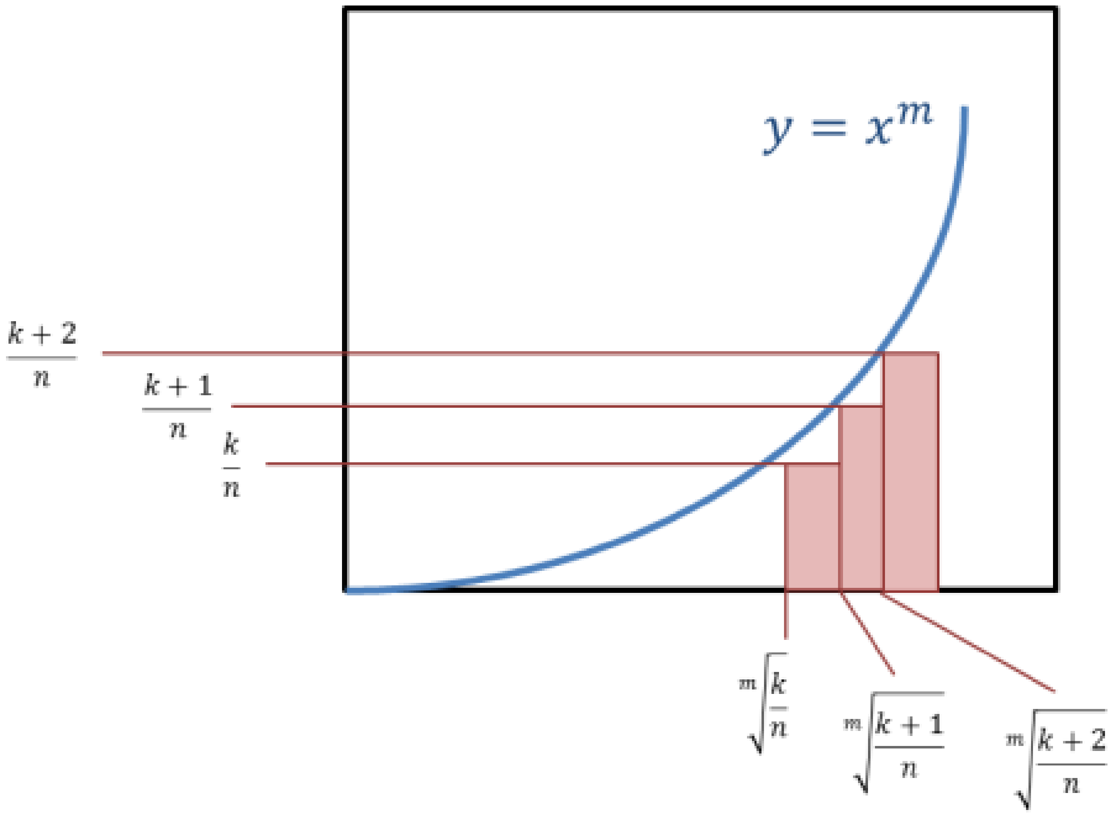
(24) SOLUTION: Let $u = \cos(x)+\sin(x)$. Then $du = (-\sin(x)+\cos(x))dx$. So $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}dx = \int_{\frac{\sqrt{3}}{2}+\frac{1}{2}}^{\sqrt{2}} \frac{1}{u}du = \ln\left(\frac{\sqrt{2}}{\frac{\sqrt{3}}{2}+\frac{1}{2}}\right) = \ln\left(\frac{2\sqrt{2}}{\sqrt{3}+1}\right)$. A.

(25) SOLUTION: $\int_1^\infty f'(\alpha x)dx = \frac{d}{d\alpha}\int_1^\infty \frac{f'(\alpha x)}{x}dx = \frac{\alpha}{\alpha^4+1}$. Let $I(\alpha) = \int_1^\infty \frac{f'(\alpha x)}{x}dx$, so that $I'(\alpha) = \frac{\alpha}{\alpha^4+1}$. Then $I(\alpha) = \int \frac{\alpha}{\alpha^4+1}d\alpha = \int \frac{\alpha}{(\alpha^2)^2+1}d\alpha = \frac{1}{2}\int \frac{1}{(u)^2+1}du = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(\alpha^2)+C$. Since $\lim_{x\to\infty} f(x) = 0$, $\lim_{\alpha\to\infty} I(\alpha) = 0$, so $\lim_{\alpha\to\infty}\left(\frac{1}{2}\arctan(\alpha^2)+C\right) = \frac{\pi}{4}+C \to C = -\frac{\pi}{4}$. Finally, $\int_1^\infty \frac{f(x)}{x}dx = I(1) = \frac{1}{2}\arctan(1) - \frac{\pi}{4} = -\frac{\pi}{8}$. B.

(26) SOLUTION: To find the volume, we use the Theorem of Pappus. $A(m) = \int_0^m \frac{4}{m^3}x(m-x)dx = \frac{4}{m^3}\int_0^m mx - x^2dx = \frac{4}{m^3}\left(\frac{m^3}{2}-\frac{m^3}{3}\right) = \frac{2}{3}$. The x-coordinate of the centroid is clearly $\bar{x} = \frac{m}{2}$ by symmetry. A student could find the y-coordinate (it is $\frac{2}{5m}$) , but all that matters is that $\lim_{m\to\infty}\bar{y} = 0$, which is obvious. The distance from the centroid to the line $0 = \frac{1}{m}x - y + 1$ is $D(m) = \frac{\left|\frac{1}{m}\bar{x}-\bar{y}+1\right|}{\sqrt{\frac{1}{m^2}+1}}$ so $\lim_{m\to\infty} D(m) = \lim_{m\to\infty}\frac{\left|\frac{1}{m}\bar{x}-\bar{y}+1\right|}{\sqrt{\frac{1}{m^2}+1}} = \lim_{m\to\infty}\frac{\left|\frac{1}{m}\cdot\frac{m}{2}-\bar{y}+1\right|}{\sqrt{\frac{1}{m^2}+1}} = \lim_{m\to\infty}\frac{\left|\frac{3}{2}-\bar{y}\right|}{\sqrt{\frac{1}{m^2}+1}} = \frac{3}{2}$. Finally, then, $V(m) = 2\pi D(m)\cdot A(m)$ so $\lim_{m\to\infty} V(m) = 2\pi\left(\frac{3}{2}\right)\left(\frac{2}{3}\right) = 2\pi$. D. Note that it is <u>not</u> intuitive that $\lim_{m\to\infty} D(m) = \frac{3}{2} \neq 1$, and you have to work it out to see that because the slope is decreasing at exactly the same rate that the x-coordinate is moving to the right, the contribution of the x-coordinate to the distance is fixed. D.

(27) SOLUTION: $P(\text{Jackson Wins}) = P(\text{Jackson flips } H) = \frac{1}{2} + \left(1-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(1-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(1-\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \cdots = \frac{1}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \cdots + \left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{n!}\right)\left(\frac{n}{n+1}\right) + \cdots = \sum_{n=1}^\infty \frac{n\left(\frac{1}{2}\right)^{n-1}}{(n+1)!} = \sum_{n=1}^\infty \frac{(n+1)\left(\frac{1}{2}\right)^{n-1}}{(n+1)!} - \sum_{n=1}^\infty \frac{\left(\frac{1}{2}\right)^{n-1}}{(n+1)!} = \sum_{n=1}^\infty \frac{\left(\frac{1}{2}\right)^{n-1}}{n!} - \sum_{n=1}^\infty \frac{\left(\frac{1}{2}\right)^{n-1}}{(n+1)!} = 2\cdot\sum_{n=1}^\infty \frac{\left(\frac{1}{2}\right)^{n}}{n!} - 4\cdot\sum_{n=1}^\infty \frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)!} = 2\left(e^{\frac{1}{2}}-1\right) - 4\left(e^{\frac{1}{2}}-1-\frac{1}{2}\right) = -2\sqrt{e} - 2 + 4 + 2 = 4 - 2\sqrt{e}$. C.

(28) SOLUTION: This sum resembles a Riemann sum, and it is, but with variable widths. In particular, as shown in the picture below, it is a Riemann sum with equal spacing along the y-axis instead:



Therefore, the sum is equal to $\int_0^1 x^m dx = \frac{1}{m+1}$. A.

(29) SOLUTION: Let $u = -x$. Then $\int_{-a}^{a} \frac{f(x)}{1+g(x)} dx = \int_{a}^{-a} \frac{f(-x)}{1+g(-x)}(-1)dx = \int_{-a}^{a} \frac{f(x)}{1+1/g(x)} dx = \int_{-a}^{a} \frac{g(x)f(x)}{g(x)+1} dx$. Therefore $2\int_{-a}^{a} \frac{f(x)}{1+g(x)} dx = \int_{-a}^{a} \frac{f(x)}{1+g(x)} dx + \int_{-a}^{a} \frac{g(x)f(x)}{g(x)+1} dx = \int_{-a}^{a} \frac{(1+g(x))f(x)}{1+g(x)} dx = \int_{-a}^{a} f(x)dx = 2K$. So $\int_{-a}^{a} \frac{f(x)}{1+g(x)} dx = K$. C.

(30) SOLUTION: $f'(x) = 4x + 3 \rightarrow f'(1) = 7$. D.