1. B
2. A
3. C
4. B
5. B
6. E
7. B
8. B
9. B
10. D
11. E
12. B
13. B
14. A
15. D
16. C
17. D
18. B
19. A
20. C
21. A
22. B
23. D
24. B
25. C
26. D
27. D
28. B
29. D
30. C

1. $\sum_{n=0}^{\infty}\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-2nx}\right)=\lim_{n\to\infty}S_n$ where $S_n=a_1+a_2+\cdots a_n$ and $a_n=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-2nx}=e^{-2n}$ (Verified with L'Hopitals). So $\sum_{n=0}^{\infty}\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-2nx}\right)=\sum_{n=0}^{\infty}e^{-2n}=\sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^n=\frac{1}{1-\left(1/e^2\right)}=\frac{e^2}{e^2-1}$.
2. Each subsequent term is found by reading out the digits in the previous term and how many there were, e.g. if $a_0=1$ then a_1 will be "one-one" ($a_1=11$) and a_2 will be read from the previous term which consists of "two-one's" ($a_2=21$). Since $a_3=1211$, the next term will be "one-one, one-two, two-ones" so the correct answer is 111221.
3. Let a_n be the number of zombie tokens you control at the beginning of turn n. Therefore, $a_1=0,a_2=0,a_3=0,a_4=0,a_5=2$. The recursion described says that $a_{n+1}=2a_n+2$. We will work backwards by replacing $a_n=2a_{n-1}+2$, so that we get $a_{n+1}=2\left(2a_{n-1}+2\right)+2=2^2\left(a_{n-1}\right)+2^2+2$. Going back another index gives $a_{n-1}=2a_{n-2}+2$ so $a_{n+1}=2^2\left(2a_{n-2}+2\right)+2^2+2=2^3a_{n-2}+2^3+2^2+2$. This process can be continued from which we have enough information to determine a new recursion, which is $a_{n+1}=2^{i+1}a_{n-i}+\sum_{j=0}^{i}2^{j+1}$ where $n-i=5$. In this instance, we want to know when $n=14$ and $i=9$. That is $a_{15}=2^{10}a_5+\sum_{j=0}^{9}2^{j+1}=2^{11}+2^{10}+\cdots+2=2^{12}-2=4096-2=4094$.
4. For each entry in the sequence there are k choices. N entries mean that there are k^n total sequences.
5. $\sum_{n=1}^{\infty}a_n=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1+2n\right)^{\frac{-4}{n}}=1$ (Try L'Hopitals)
6. Using the root test we get $\lim_{n\to\infty}\left|\frac{\left(x-3\right)^{2n}}{n^n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{\left(x-3\right)^2}{n}\right|=0$ therefore, the value of x does not affect convergence. The I.o.C. is $\left(-\infty,\infty\right)$.
7. Trying the traditional technique of evaluating the limit of a recursive sequence fails here. Alternatively, if you attempt to start evaluating the next few terms of the sequence, you will see the following pattern:

$a_2 = 1+\frac{1}{2}, a_3 = 1+\frac{1}{2}+\frac{1}{2^2}, a_4 = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}$ and in general $a_n = \sum_{i=0}^{n-1}\left(\frac{1}{2}\right)^i$ so that

$\lim_{n\to\infty} a_n$ is really the sum of the geometric series $\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = \frac{1}{1-\frac{1}{2}} = 2$.

8. A) False, counterexample is $a_n = \frac{1}{n}$

   B) True, test for divergence.

   C) False, counterexample $a_n = 1, b_n = -1$

   D) True, in the sequence of partial sums for $a_n + \frac{1}{n}$ we will get close to the sum of a_n plus the sum of $\frac{1}{n}$ which is divergent, therefore the overall sum is divergent.

9. Telescoping Sum – breaking up into four terms allows us to more easily find the nth partial sum

$$\sum_{n=1}^{\infty}\left(\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+1}\right)+\cos\left(\frac{1}{n+2}\right)-\cos\left(\frac{1}{n+1}\right)\right)$$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+1}\right)+\sum_{n=1}^{\infty}\cos\left(\frac{1}{n+2}\right)-\cos\left(\frac{1}{n+1}\right)$$

$$=\left(\cos(1)-\cos\left(\frac{1}{2}\right)+\cos\left(\frac{1}{2}\right)-\cos\left(\frac{1}{3}\right)+\ldots+\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+1}\right)\right)$$

$$+\left(\cos\left(\frac{1}{3}\right)-\cos\left(\frac{1}{2}\right)+\cos\left(\frac{1}{4}\right)-\cos\left(\frac{1}{3}\right)+\cos\left(\frac{1}{5}\right)-\cos\left(\frac{1}{4}\right)+\cos\left(\frac{1}{n+2}\right)-\cos\left(\frac{1}{n+1}\right)\right)$$

Seeing which terms will cancel gives the nth partial sum as

$S_n = \cos(1)-\cos\left(\frac{1}{2}\right)+\cos\left(\frac{1}{n+2}\right)-\cos\left(\frac{1}{n+1}\right)$ from which we determine the sum by taking $\lim_{n\to\infty} S_n = \cos(1)-\cos\left(\frac{1}{2}\right)$

10. The Alternating Series Estimation Theorem says that for a convergent alternating series the difference between the actual sum, s, and the nth partial sum, S_n, is less than or equal to the next term in the partial sum. That is, $\left|s - S_n\right| £ b_{n+1}$. In this case, $b_n = \frac{1}{n(n+1)}$. We want $b_{n+1} £ \frac{1}{49}$, that is $\frac{1}{(n+1)(n+2)} \le \frac{1}{49}$. The smallest such n is $n=6$.

11. A) Combining logarithms gives $\left\{\ln(2n)-\ln(n+1)\right\}=\left\{\ln\left(\frac{2n}{n+1}\right)\right\}$.

$\lim\limits_{n\to\infty}\ln\left(\frac{2n}{n+1}\right)=\ln\left(\lim\limits_{n\to\infty}\frac{2n}{n+1}\right)=\ln 2$. Convergent.

B) $\lim\limits_{n\to\infty}\frac{\cos\left(\sqrt{n}\right)}{n^2}=0$ by the Squeeze theorem. Convergent.

C) Squeeze theorem has this limit going to zero. Convergent.

D) 3^n has a faster growth rate than n^{100} therefore the limit is infinity. Divergent.

E) n^n has a faster growth rate than 2^n therefore the limit is 0. Convergent.

12. A) Integral test shows divergent.

B) Comparison test with $\sum\limits_{n=3}^{\infty}\frac{1}{n^5}$ shows convergent.

C) Alternating Series Test shows convergent.

D) Integral Test shows divergent.

13. This is a recursive sequence such that $a_{n+1}=\sqrt{5a_n}$. By the Monotonic Sequence Theorem, increasing and bounded above implies convergence. Solving algebraically we have $L=\sqrt{5L}$ from which we get $L=5$ and $L=0$ of which only $L=5$ is correct.

14. Since $\sum\limits_{n=1}^{\infty} a_n = 4$, it follows that $L=\lim\limits_{n\to\infty}a_n=0$ and $s=\lim\limits_{n\to\infty}S_n=4$. So $L+s=4$.

15. You can approach this either as asking for the interval of convergence of a power series, or as finding the common ratio of a convergent geometric series. Using the latter approach we have that $r=\frac{5x-2}{3}$, and will be convergent if $\left|\frac{5x-2}{3}\right|<1$. Solving that inequality gives, $-\frac{1}{5}<x<1$.

16. Looking at the highest order terms of the numerator and denominator gives an appropriate comparison of $\sum\frac{n^3}{\sqrt[4]{n^k}}=\sum\frac{1}{n^{k/4-3}}$ which is a convergent p series if $k/4-3>1$ or if $k>16$. Limit /Direct Comparison will show that $\sum\limits_{n=5}^{\infty}\frac{n^3+2n^2}{\sqrt[4]{n^k}-1}$ converges when $\sum\frac{n^3}{\sqrt[4]{n^k}}=\sum\frac{1}{n^{k/4-3}}$ converges.

17. Looking at the summation it is of the form $S=\frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\frac{16}{16}+\frac{25}{32}+\ldots$. Furthermore we have $S-\frac{S}{2}=\frac{S}{2}=\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}+\ldots$ Continuing this process again we have $\frac{S}{2}-\frac{S}{4}=\frac{S}{4}=\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$. This is a geometric series and we have $\frac{S}{4}=\frac{1}{2}+\frac{1/2}{1-1/2} \Rightarrow S=4(3/2)=6$. D

18. a) Limit comparison with $b_n=\frac{1}{n^2}$ shows it is convergent.

    b) Limit comparison with $b_n=\frac{1}{n}$ shows it is divergent.

    c) Limit comparison with $b_n=\frac{1}{n^3}$ shows it is convergent.

    d) Test for Divergence shows divergent.

    e) Limit comparison with $b_n=\frac{1}{n^2}$ shows it is convergent.

19. The initial attempt at finding the coefficients of the Taylor series will be problematic with the tedious derivatives. Instead we will use the MacLaurin series and substitute in after completing the square, which allows our Taylor series to be centered at 3/2.

$$\ln\left(4x^2-12x+13\right)=\ln\left(4\left(x-\frac{3}{2}\right)^2+4\right)=\ln(4)+\ln\left(1+\left(x-\frac{3}{2}\right)^2\right)$$

$$=\ln(4)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(x-\frac{3}{2}\right)^{2n}}{n}$$

20. First we must get a power series for the fraction $\frac{e^{t^2}-1-t^2}{t^4}$. The series for $e^{t^2}=\sum_{n=0}^{\infty}\frac{t^{2n}}{n!}=1+t^2+\cdots$ which means $e^{t^2}-1-t^2=\sum_{n=2}^{\infty}\frac{t^{2n}}{n!}$ and finally $\frac{e^{t^2}-1-t^2}{t^4}=\sum_{n=2}^{\infty}\frac{t^{2n-4}}{n!}$. Integrating gives $\int_0^x\sum_{n=2}^{\infty}\frac{t^{2n-4}}{n!}dt=\sum_{n=2}^{\infty}\frac{x^{2n-3}}{n!(2n-3)}$.

21. Begin by finding the MacLaurin series for $\sin\left(\frac{x^2}{2}\right)$ which we get from substitution into the known series: $\sin\left(\frac{x^2}{2}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2^{2n+1}(2n+1)!}$. Then figure out what value of n will gives us 42, in this case $4n+2=42$ so $n=10$. The only term

that does not go to zero (either by differentiation, or by evaluation at x=0) will be when n=10, therefore we only care about $\frac{(-1)^{10}x^{42}}{2^{21}21!}$. Differentiating 42 times gives $\frac{42!}{2^{21}21!}$.

22. Looking at the denominators, the only function that has odd factorial values is $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$. Looking at the numerators, those are powers of 2 so we can make that guess that $x = 2$. Exploring that idea, we see $\sin(2) = 2 - \frac{8}{3!} + \frac{32}{5!} - \cdots$. Manipulating terms it becomes clear that $\frac{8}{3!} - \frac{32}{5!} + \frac{128}{7!} - \cdots = 2 - \sin 2$.

23. Start with the function/power series $\frac{1}{3-x} = \frac{1/3}{1-x/3} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$. We can differentiate the function, and the corresponding power series to get $\frac{1}{(3-x)^2} = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{3^{n+1}}$. Multiplying by x^3 gives
$$\frac{x^3}{(3-x)^2} = x^3\sum_{n=1}^{\infty}\frac{nx^{n-1}}{3^{n+1}} = \sum_{n=1}^{\infty}\frac{nx^{n+2}}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{(n+1)x^{n+3}}{3^{n+2}}.$$

24. This is the MacLaurin series for cosine evaluated at a particular value. First rewrite as $\frac{1}{3}\sum_{n=2}^{\infty}\frac{(-1)^n\pi^{2n}}{3^{2n}(2n)!} = \frac{1}{3}\sum_{n=2}^{\infty}\frac{(-1)^n(\pi/3)^{2n}}{(2n)!}$ where $\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/3)^{2n}}{(2n)!} = \cos\frac{\pi}{3}$, so this just becomes $\frac{1}{3}\left(\cos\left(\frac{\pi}{3}\right) - 1 + \frac{\pi^2}{18}\right) = \frac{1}{3}\left(\frac{1}{2} - 1 + \frac{\pi^2}{18}\right) = \frac{\pi^2}{54} - \frac{1}{6}$.

25. The sum of squares of the first ten positive even integers is $\sum_{n=1}^{10}(2n)^2 = 4\sum_{n=1}^{10}n^2 = 4\frac{(10)(11)(21)}{6} = 1540$.

26. Since this is a convergent geometric sum the difference between the actual sum $\sum_{n=0}^{\infty}\left(\frac{3}{10}\right)^n$ and the approximation after 100 terms $\sum_{n=0}^{99}\left(\frac{3}{10}\right)^n$ is
$$\sum_{n=0}^{\infty}\left(\frac{3}{10}\right)^n - \sum_{n=0}^{99}\left(\frac{3}{10}\right)^n = \sum_{n=100}^{\infty}\left(\frac{3}{10}\right)^n = \frac{(3/10)^{100}}{1-(3/10)} = \frac{3^{100}}{10^{99}\cdot 7}.$$

27. We are given enough information to construct an interval of convergence as well as intervals for divergence. The series is centered at $x = 2$ and we are told that the series is convergent for $x = 4$. Since the interval must be symmetric around

the center, and we went 2 units to the right, we may also move 2 units to the left to arrive at $x = 0$, although anytime we use symmetry to get the other endpoint, we do not have enough information to determine what is happening there. So far we know the series is convergent for $0 < x \le 4$. Likewise, we are told the series is divergent at $x = -2$, a distance of 4 from the center. This also means anything to the left of $x = -2$ is divergent. We can use symmetry to go 4 units to the right of the center and arrive at $x = 6$. We don't have enough information at this point, but anything to the right will be divergent. So we have that for $x \le -2, x > 6$ the series is divergent and we don't have enough information at $x = 6$. Any value not on the given intervals, we do not have enough information about. For the answer choices, we must plug in $x = 0$, $x = 6$, $x = 5$, $x = 1$ respectively. The only value we have information about is $x = 1$ which is inside the interval of convergence. Therefore, the series is convergent here. D

28. Evaluating the inner sum as follows: $1 + i\pi - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + .. = \sum_{n=0}^{\infty} \frac{(i\pi)^n}{n!} = e^{i\pi}$, therefore we have a final answer of $\ln\left(e^{i\pi}\right) = i\pi$.

29. Sum = first term/ (1-ratio), based on what we are given: $3 - i = \frac{1+i}{1-r}$ and upon rearranging we have $1 - r = \frac{1+i}{3-i} \cdot \frac{3+i}{3+i} = \frac{2+4i}{10}$. Solving for r yields $r = \frac{8-4i}{10} = \frac{4-2i}{5}$.

30. This is a Riemann sum that reduces to $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}$. C