ANSWERS

1. C
2. C
3. D
4. B
5. D
6. D
7. A
8. B
9. E (0)
10. E
11. A
12. C
13. C
14. B
15. E (1135)
16. C
17. B
18. E (879)
19. B
20. B
21. C
22. D
23. D
24. D
25. A
26. E (313)
27. A
28. A
29. C
30. D

SOLUTIONS

1. The number 18 in base $b$ translates to $b + 8$ and we need this to equal 21. Hence $b = 13$.

2. The prime factorization of 15! is $2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, so the number of factors it has is $(11+1)(6+1)(3+1)(2+1)(1+1)(1+1) = (12)(7)(4)(3)(2)(2) = 4032$.

3. Let $L$ and $W$ be the length and width. Then $LW = 6(L + W)$, or $(L - 6)(W - 6) = 36$. Since there are 5 distinct ways of factoring 36, there are 5 different rectangles.

4. The largest integer that cannot be made using multiples of 8 and 15 is $8 \times 15 - 8 - 15 = 97$.

5. We factor $3^{12} - 1$ into $(3^6 + 1)(3^3 + 1)(3^3 - 1) = (730)(28)(26) = (73)(13)(7)(5)(2^4)$, so we see the largest prime factor is 73.

6. Since $2017 = 5 \cdot 343 + 6 \cdot 49 + 7 + 1$, the base-7 representation of 2017 is $5611_7$. The average of the digits is therefore $\frac{5+6+1+1}{4} = \frac{13}{4} = 3.25$.

7. Let $N = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$ be a scrunchy number, where each $a_i$ is a single digit. Clearly, $k > 1$. The sum of the digits of $N$ is $S = a_k + a_{k-1} + \cdots + a_0$ and the product is $P = a_k \cdot a_{k-1} \cdot \cdots \cdot a_0$. Then $P \leq 9^k a_k$, with equality if and only if all digits are 9. Note that $N - S - P = (10^k - 1)a_k + (10^{k-1} - 1)a_{k-1} + \cdots + (10 - 1)a_1 - P \geq (10^k - 1)a_k - 9^k a_k = (10^k - 9^k - 1)a_k > 0$. Equality holds if and only if $k = 1$ and $a_0 = 9$. Hence the only scrunchy numbers are 19, 29, 39, 49, 59, 69, 79, 89, and 99. There are 9 of them.

8. Consider the two children with the smallest total of cards made. Suppose this total is at most 8. Since any group of three children made no less than 14, everyone else makes at least 6. But since everyone else makes 6, there are groups of five that make more than 25. It follows that the smallest is at least 9. If it is at least 10, then everyone made 5 postcards, for a total of 85, which is clearly not even. Hence the smallest total is 9, with two children making 4 and 5 postcards, respectively. It follows that every other child made 5 postcards, for a total of 84.

9. Let $n = 10t + u$ be such a number. Then since interchanging the digits doubles the number, we must have $10t + u = 2(10u + t)$, or $8t = 19u$, but this is impossible. Thus, there are no such numbers.

10. The digits of 20142018 add up to 18, so this is divisible by 9. The number 20172017 is clearly divisible by 2017. The digits of 20172018 add up to 21, so this is divisible by 3. The alternating digit sum of 20172020 is 0 so this is divisible by 11. This leaves 20182021 as the prime number (and it is, in fact, prime).

11. Note that the sum of two primes can only be prime if one of the two primes is 2. Since $2999 + 2 = 3001$ and 2999 and 3001 are each prime, we have $2999 + 2 + 3001 = 6002$.

12. 7 types of coins suffice: for example, coins worth 1, 2, 4, 8, 16, 32 and 64. Fewer coins cannot work since with $n$ coins there is at most $2^n$ combinations where each coin is used 0 or 1 times; and $2^6 = 64 < 100$.

13. We have $4^3 \cdot 6^2 \cdot 12^5 = 2^6 \cdot 2^2 \cdot 3^2 \cdot 2^{10} \cdot 3^5 = 2^{10} \cdot 3^2 \cdot 2^8 \cdot 3^5$ so that $k^2 = 2^{10} \cdot 3^2 = (2^5 \cdot 3)^2 = 96^2$. Hence $k = 96$.

14. The possible factorizations of 24 into three positive integers are 1, 1, 24; 1, 2, 12; 1, 3, 8; 1, 4, 6; 2, 3, 4; and 2, 2, 6. The possible sums of these factorizations are 26, 15, 12, 11, 9, and 10. There are 6 possible sums.

15. Since 275 factors into $5^2$ and 11, we need at minimum another factor of 5 and two factors of 11 to make a perfect cube. Hence $N = 5 \cdot 11^2 = 605$. Noting that $605 = 512 + 64 + 3 \cdot 8 + 5$ gives $1135_8$ as the octal representation of 605.

16. Suppose the original integer (before squaring) has tens digit $t$ and units digit $u$. Then $(10t + u)^2 = 100t^2 + 20tu + u^2$. However, the carry from $u^2$ must be odd since 7 is odd. Thus $u$ must be either 4 or 6. In either case, the units digit of the square is 6.

17. The number 27001 can be written as $30^3 + 1 = (30 + 1)(30^2 - 30 + 1) = (31)(871)$. But 871 is not prime; its factors are 67 and 13, so the largest prime factor is 67.

18. Since 2017 is prime, we have, by Fermat's Little Theorem, $17^{2019} \equiv 17^3 \pmod{2017}$. This reduces modulo 2017 from $17^3 = 4913$ to 879.

19. The factors of 18 are 1, 2, 3, 6, 9, and 18. Clearly, 1 is a quadratic residue. We use the Legendre Symbol and Gauss' Law of Quadratic Reciprocity to check the others.

$$\left(\frac{2}{19}\right) = (-1)^{(19^2-1)/8}\left(\frac{19}{2}\right) = (-1)^{45}\left(\frac{1}{2}\right) = (-1)(1) = -1$$
$$\left(\frac{3}{19}\right) = (-1)^{(19-1)(3-1)/4}\left(\frac{19}{3}\right) = (-1)^{9}\left(\frac{1}{3}\right) = (-1)(1) = -1$$
$$\left(\frac{6}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{3}{19}\right) = (-1)(-1) = 1$$
$$\left(\frac{9}{19}\right) = \left(\frac{3}{19}\right)\left(\frac{3}{19}\right) = (-1)(-1) = 1$$
$$\left(\frac{18}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{9}{19}\right) = (-1)(1) = -1$$

Hence, only 1, 6, and 9 are quadratic residues modulo 19, and so there are three.

20. We factor: $28! + 29! + 30! = 28!\,(1 + 29 + 29 \cdot 30) = 28!\,(30^2)$. The largest prime factor of this number is clearly 23, but there is only of them. The largest prime which appears twice as a factor is 13 – once in 13 and again in 26.

21. To obtain the smallest value, we should make $x$ be 2 since $x$ has an exponent. Then, $y$ should be 3 (recall they are distinct primes). Finally, the smallest prime not used is 5, so $z$ should be 5. Therefore $x^y + z = 2^3 + 5 = 13$.

22. Translate each of the congruences into an equation. We have, for some integers $t$, $s$, and $u$, $x = 10 + 13t$, $x = 3 + 9s$, and $x = 5 + 7u$. Substitute the first equation into the second, modulo 9, to get $10 + 13t \equiv 3 \pmod 9$. This becomes $t \equiv 5 \pmod 9$, or $t = 5 + 9s$. Now substitute this into the first equation. We get $x = 10 + 13(5 + 9s) = 75 + 117s$. Finally, substitute this into the third equation, modulo 7, to get $75 + 117s \equiv 5 \pmod 7$. This reduces to $5s \equiv 0 \pmod 7$ so that $s = 7u$. Now we get $x = 75 + 117(7u) = 75 + 819u$. Thus the positive solutions are the set of integers $\{75, 894, 1713, \dots\}$ and the smallest is 75. The sum of the digits is 12.

23. Both $P$ and $Q$ have an odd number of divisors, so they are each perfect squares. Since the square of a prime has three divisors, we seek the square of a composite number. Checking the first few such squares, we find that 16 has five divisors; 36 has nine divisors, and 64 has seven divisors. Hence, $P$ must be 64 and $Q$ must be 36, and their sum is 100.

24. Let's compute the Fibonacci numbers modulo 5 and look for a pattern. We have

$$1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, \dots$$

The pattern is periodic with period 20. Hence, the 2018th (since we started with index 0) Fibonacci number has the same remainder modulo 5 as the 18th Fibonacci number, which is 4.

25. The left-hand side of $1! + 2! + 3! + \cdots + x! = y^z$ is divisible 3, so the right-hand side is divisible by 3. This implies the right-hand side is a multiple of $3^z$. But when $x \geq 8$, we have $1! + 2! + 3! + \cdots + 8! = 46233$, which is divisible by 9, but not by 27; and thus, the right-hand side is divisible by 9, but not by 27. Therefore $z = 2$. Now, taking the equation modulo 5, we have $1! + 2! + 3! + \cdots + x! = 1! + 2! + 3! + 4! \equiv 3 \pmod 5$. On the other hand, $y^2$ is congruent to either 0, 1, or 4 modulo 5. Thus there are no solution when $x \geq 8$. Now for $x < 8$, we simply check the possibilities to find the only solution: $1! + 2! + 3! = 3^2$. Hence, we have $x = y = 3$ and $z = 2$ so that $x + y + z = 8$.

26. We use the Euclidean algorithm to find each greatest common divisor. 7592 divided by 5913 is 1 with remainder 1679; 5913 divided by 1679 is 3 with remainder 876; 1679 divided by 876 is 1 with remainder 803; 876 divided by 803 is 1 with remainder 73; 803 divided by 73 is 11 with remainder zero. Hence, $G$ is 73. Next, 851 divided by 481 is 1 with remainder 370; 481 divided by 370 is 1 with remainder 111; 370 divided by 111 is 3 with remainder 37; 111 divided by 37 is 3 with remainder zero. Hence, $C$ is 37. Finally, 3248 divided by 1827 is 1 with remainder 1421; 1827 divided by 1421 is 1 with remainder 406; 1421 divided by 406 is 3 with remainder 203; 406 divided by 203 is 2 with remainder zero. Hence, $D$ is 203. Thus, $G + C + D = 73 + 37 + 203 = 313$.

27. Since $1184 = 2^5 \cdot 37$, we find the sum of the proper divisors to be $(1 + 2 + 4 + 8 + 16 + 32)(1 + 37) - 1184 = (63)(38) - 1184 = 1210$.

28. Let $x$ be the number of fruits in a pile. Let $y$ be the number each receives. Then $63x + 7 = 23y$. The smallest positive solution is $x = 5$ and $y = 14$. So there were 5 plantains in each heap.

29. We have $3 = 1 + 2$ (where 1 is the power of 2 and 2 is the prime); $119 = 16 + 103$; $723 = 512 + 211$; and $1015 = 512 + 503$. The number 509 cannot be written as the sum of a power of 2 and a prime.

30. Let $N = n_k 9^k + n_{k-1} 9^{k-1} + \cdots + n_1 9 + n_0 = t_k 13^k + t_{k-1} 13^{k-1} + \cdots + t_1 13 + t_0$. Clearly the digits $n_i$ must be nonnegative and must be less than 9, while the digits $t_i$ may be any nonnegative digits. Since $N$ is a 9-13-base-double, we know the base-9 representation is twice the base-13 representation; that is,

$$n_k 10^k + n_{k-1} 10^{k-1} + \cdots + n_1 10 + n_0 = 2(t_k 10^k + t_{k-1} 10^{k-1} + \cdots + t_1 10 + t_0).$$

Now, suppose $k = 3$. Then $N = 729n_3 + 81n_2 + 9n_1 + n_0 = 2197t_3 + 169t_2 + 13t_1 + t_0$ and $1000n_3 + 100n_2 + 10n_1 + n_0 = 2000t_3 + 200t_2 + 20t_1 + 2t_0$. Subtracting these lines from each other yields $271n_3 + 19n_2 + n_1 = -197t_3 + 31t_2 + 7t_1 + t_0$. Choosing $n_3 = 2$ and $t_3 = 1$, the least values these could be for four-digit numerals, the equation becomes $739 + 19n_2 + n_1 = 31t_2 + 7t_1 + t_0$, since the right-hand side has maximum value 351 while the left-hand side has minimum value 739. Choosing $k$ greater than 3 widens the discrepancy between these values.

Therefore, let's try $k = 2$. Then $N = 81n_2 + 9n_1 + n_0 = 169t_2 + 13t_1 + t_0$ and $100n_2 + 10n_1 + n_0 = 200t_2 + 20t_1 + 2t_0$. Subtracting these lines yields $19n_2 + n_1 = 31t_2 + 7t_1 + t_0$. To make this as large as possible, given the previously mentioned constraints, setting $n_2 = 8$ and $t_2 = 4$ yields $28 + n_1 = 7t_1 + t_0$ . The only possible values for $t_1$ are 3, 4, or 5, but these values would make $n_1$ 6 or 7 (too large); 8 (which would make $t_0 = 8$, meaning you must carry, except that 8 is exactly twice 4, so no carrying can be made); or 0 or 1 (too small), respectively.

Therefore, we will try the next largest combination, setting $n_2 = 7$ and $t_2 = 3$ yields $40 + n_1 = 7t_1 + t_0$. The only possible values for $t_1$ are 5, 6, or 7, and these values would make $n_1$ 0 or 1; 2 or 3; or 4 or 5 (these two are too small), respectively. Therefore, working from the top down, let us consider the case where $t_1 = 6$ and $n_1 = 3$. These values imply $t_0 = 1$, which means there would be no numbers to carry from the ones place (which the combination of $t_1 = 6$ and $n_1 = 3$ suggests must take place). Therefore, we will test $t_1 = 6$ and $n_1 = 2$, which imply $t_0 = 0$ and $n_0 = 0$, and all relationships are consistent.

Therefore, $720_9 = 360_{13} = 585_{10}$, making 585 the greatest 9-13-base-double, the sum of whose digits is 18.