Answers:

Section I: Easy Problems

1) 40
2) 56
3) 460
4) 3:15 pm
5) 16
6) 25
7) 1.75
8) 23
9) 9:36 am
10) 1

Section II: Hard Problems

1) (a) ln2
   (b) lnln2
   (c) there is no solution
2) $\times$, $\div$, $+$
3) forty
4) 508
5) 2
6) $(14,520)$
7) 18
8) $(1,2,3)$ and $(2,3,7)$
9) D, G, and H
10) $\frac{-1+\sqrt{5}}{2}$

Section III: 1, 2, 3, 4

1) $\frac{10}{81}$, $\frac{29}{235}$, $\frac{39}{316}$, $\frac{41}{332}$, and $\frac{49}{397}$
2) 13807
3) 1003
4) 854

Section IV: I Suspect You Are Irrational!

1) $\pi$
2) $e$

3) $\phi$ or the golden ratio
4) Euler's constant or Euler-Mascheroni constant
5) plastic constant OR plastic number OR minimal Pisot number
6) Sierpinski's constant
7) reciprocal Fibonacci constant
8) Gauss' constant
9) omega constant
10) universal parabolic constant
11) Erdos-Borwein constant
12) Ramanujan-Soldner constant OR Soldner constant

Section V: Simple Graphs, Ridiculous Equations

1) $(3,-4)$
2) $(3,-4)$
3) $-2/3$ (the lines are $2x+3y+2=0$ and $2x+3y-1=0$)
4) $(5,3)$
5) $(-1,2)$ and $(3,7)$
6) $(-3,5)$ and $(5,-3)$
7) $1/5$ (the point is $(1,-1)$ and the line is $-3x+4y+6=0$)
8) $(0,2)$ (the three lines are $3x+y-2=0$, $-x+3y-6=0$, and $-2x-y+2=0$)
9) 6 (the three points are $(-2,1)$, $(2,1)$, and $(-4,-2)$)
10) $(3,-2)$

Section VI: Cryptograms

Instructions: PLEASE DECIPHER THESE. ALSO, CONGRATULATIONS TO YOU FOR DECIPHERING THESE INSTRUCTIONS, ALTHOUGH THEY ARE NOT WORTH POINTS. IN BLANK NUMBER THIRTEEN, WRITE "THANK YOU". BRING YOUR ANSWER SHEET UP TO DR. KUSTOS TO CHECK, BUT HURRY—ONLY THE FIRST TEN TEAMS TO DO SO WILL EARN BONUS POINTS…
1) EVERY POSITIVE INTEGER CAN BE WRITTEN AS THE SUM OF AT MOST FOUR PERFECT SQUARES.
2) CONGRUENT PARTS OF CONGRUENT TRIANGLES ARE CONGRUENT.
3) IN ANY SEPARATION OF A PLANE INTO CONTIGUOUS REGIONS, NO MORE THAN FOUR COLORS ARE REQUIRED TO COLOR THE REGIONS TO THAT NO TWO ADJACENT REGIONS HAVE THE SAME COLOR.
4) EVERY NON-CONSTANT SINGLE-VARIABLE POLYNOMIAL WITH COMPLEX COEFFICIENTS HAS AT LEAST ONE COMPLEX ROOT.

5) ANY CONTINUOUS, REAL-VALUED FUNCTION ON A CLOSED AND BOUNDED INTERVAL MUST ATTAIN A MAXIMUM VALUE AND A MINIMUM VALUE AT SOME POINTS ON THAT INTERVAL.
6) THERE EXIST INFINITELY MANY PRIME NUMBERS.
7) IF AN INFINITE SERIES IS CONDITIONALLY CONVERGENT, THEN ITS TERMS CAN BE REARRANGED SO THAT THE NEW SERIES EITHER CONVERGES TO ANY GIVEN VALUE OR DIVERGES.
8) EVERY EVEN PERFECT NUMBER IS THE PRODUCT OF A POWER OF TWO AND A MERSENNE PRIME.
9) FOR ANY SET A, THE POWER SET OF A HAS A STRICTLY GREATER CARDINALITY THAN THE CARDINALITY OF A.
10) A TO THE P MINUS ONE IS CONGRUENT TO ONE MODULO P IF A AND P ARE RELATIVELY PRIME.
11) A TO THE N PLUS B TO THE N EQUALS C TO THE N HAS NO POSITIVE INTEGER SOLUTIONS FOR A, B, AND C WHEN N IS AN INTEGER GREATER THAN OR EQUAL TO THREE.
12) FOR ANY FINITE GROUP G, THE ORDER OF EVERY SUBGROUP OF G DIVIDES THE ORDER OF G.

Section VII: X-treme Sudoku

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 12 | 8 | 10 | 11 | 14 | 4 | 7 | 6 | 13 | 15 | 16 | 2 | 5 | 1 | 3 |
| 13 | 16 | 5 | 7 | 15 | 2 | 1 | 3 | 11 | 9 | 10 | 4 | 12 | 8 | 14 | 6 |
| 4 | 1 | 6 | 14 | 9 | 16 | 8 | 10 | 3 | 2 | 5 | 12 | 7 | 13 | 15 | 11 |
| 11 | 2 | 15 | 3 | 12 | 5 | 6 | 13 | 7 | 14 | 1 | 8 | 4 | 9 | 16 | 10 |
| 5 | 11 | 9 | 6 | 7 | 13 | 3 | 16 | 15 | 4 | 12 | 1 | 10 | 2 | 8 | 14 |
| 3 | 8 | 7 | 4 | 10 | 12 | 2 | 15 | 14 | 5 | 16 | 13 | 1 | 11 | 6 | 9 |
| 1 | 13 | 10 | 15 | 14 | 6 | 11 | 5 | 8 | 7 | 9 | 2 | 3 | 16 | 12 | 4 |
| 12 | 14 | 2 | 16 | 1 | 8 | 9 | 4 | 10 | 6 | 3 | 11 | 13 | 15 | 5 | 7 |
| 8 | 3 | 1 | 12 | 16 | 7 | 14 | 6 | 2 | 15 | 4 | 5 | 9 | 10 | 11 | 13 |
| 16 | 10 | 11 | 2 | 3 | 1 | 5 | 12 | 9 | 8 | 13 | 7 | 6 | 14 | 4 | 15 |
| 6 | 7 | 13 | 5 | 4 | 15 | 10 | 9 | 1 | 11 | 14 | 3 | 16 | 12 | 2 | 8 |
| 15 | 4 | 14 | 9 | 8 | 11 | 13 | 2 | 16 | 12 | 6 | 10 | 5 | 3 | 7 | 1 |
| 10 | 5 | 4 | 11 | 6 | 9 | 16 | 8 | 13 | 1 | 2 | 14 | 15 | 7 | 3 | 12 |
| 7 | 6 | 12 | 13 | 5 | 3 | 15 | 14 | 4 | 16 | 11 | 9 | 8 | 1 | 10 | 2 |
| 14 | 9 | 3 | 8 | 2 | 4 | 12 | 1 | 5 | 10 | 7 | 15 | 11 | 6 | 13 | 16 |
| 2 | 15 | 16 | 1 | 13 | 10 | 7 | 11 | 12 | 3 | 8 | 6 | 14 | 4 | 9 | 5 |

Section VIII: You Must Be Joking, My Head Hurts Already

1) 288
2) 6,670,903,752,021,072,936,960
3) (3,4,5), (20,21,29), (119,120,169), (696,697,985), (4059,4060,5741), (23660,23661,33461), (137903,137904,195025), (803760,803761,1136689)
4) (3,4,5), (13,14,15), (51, 52, 53), (193,194,195), (723,724,725), (2701,2702,2703), (10083,10084,10085), (37633,37634,37635), (140451,140452,140453), (524173,524174,524175)

5) $x = \frac{5}{2 - \sqrt[3]{9}}$ or $x = -20 - 15\sqrt[3]{3} - 10\sqrt[3]{9}$

6) 7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79, 87, 92, 95, 103, 111, 112, 119, 124, 127, 135, 143, 151, 156, 159, 167, 175, 183, 188, 191, 199

7) real roots: $-4,\ -3,\ -3,\ -2,\ -1,\ 1,\ 1,\ 3,\ \frac{5 \pm \sqrt{13}}{2},\ 2 \pm \sqrt{3}$

imaginary roots: $2 \pm i,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Solutions to selected problems:

Section I: Easy Problems

1) $0.50x+3(50-x)=50 \Rightarrow 150-2.50x=50 \Rightarrow 2.50x=100 \Rightarrow x=40$

2) $\frac{324}{36}=9$ full shelves of novels, so 4 shelves are for encyclopedias $\Rightarrow 14\cdot 4=56$ encyclopedias maximum

3) $\frac{1600+x}{4}=x+55 \Rightarrow 1600+x=4x+220 \Rightarrow 3x=1380 \Rightarrow x=460$

4) The watch advances 80% of the actual elapsed time, so if the watch has advanced 5 hours (300 minutes), $\frac{300}{x}=\frac{4}{5} \Rightarrow x=375$, so the actual elapsed time is 375 minutes, or 6 hours and 15 minutes. Therefore, the actual time is 3:15 pm.

5) If 31 members are singers, 23 members are dancers. Since there are 17 female dancers, there are 6 male dancers. Since there are 22 male members, there are 16 male singers.

6) Burt's and Ernestine's ages are always 24 years apart, so $\frac{1}{3}=\frac{37-x}{61-x} \Rightarrow 111-3x=61-x$ $\Rightarrow 2x=50 \Rightarrow x=25$.

7) Let $r$ and $c$ be Tracy's and the current's rates, respectively, in miles per hour. Then $\frac{1}{2}(r+c)=3$ $r+c=6$ and $\frac{6}{5}(r-c)=3 \Rightarrow r-c=\frac{5}{2}$. Solving this system, $c=\frac{7}{4}$.

8) Let $M$ and $D$ be the number of votes that Mickey and Donald received, respectively. Then $M-7=D+7 \Rightarrow M=D+14$ (and based on this, Mickey won the election). Further, $M+5=7(D-5) \Rightarrow M=7D-40$. Therefore, $D+14=7D-40 \Rightarrow 6D=54 \Rightarrow D=9 \Rightarrow M=23$. So 23 votes won the election.

9) Let $x$ be the number of minutes that have elapsed since midnight. Since there are 1440 minutes in a day, $\frac{x}{4}=\frac{1440-x}{6} \Rightarrow 5760-4x=6x \Rightarrow 5760=10x \Rightarrow x=576$. Therefore, it is 576 minutes after midnight, or 9:36 am.

10) Since the blue cup and orange lid go together, the yellow cup must go with the red lid, and the green cup must go with the yellow lid. Since the yellow cup and red lid go together, neither straw that is that color can go with it, so those go with the blue straw. Of what is remaining, the green cup and yellow lid must go with the red straw (otherwise, you would have to pair the yellow lid and straw together). Therefore, the blue cup and orange lid go with the yellow straw, thus meaning that there is only one possible configuration of the drinking cup setups.

Section II: Hard Problems

1) (c) There is no solution to this because $0<\ln 2<1$, meaning that $\ln\ln 2<0$. Since the answer is supposed to be $\ln\ln\ln 2$, this would be the natural logarithm of a negative number, which is impossible.

4) Let $a$ and $b$ be the first two terms of the sequence. Then the third through tenth terms are, in order, $a+b$, $a+2b$, $2a+3b$, $3a+5b$, $5a+8b$, $8a+13b$, $13a+21b$, and $21a+34b$. Therefore, $5a+8b=120$, meaning that $b$ must be a multiple of 5 since 120 and 5$a$ are while 8 is not. The only reasonable choices to fit this equation are $b=5$ or $b=10$, but $b=5$ would make $a=16$, violating that this is an increasing sequence. Therefore, $b=10 \Rightarrow a=8$, thus making $a_{10}=21(8)+34(10)=508$ (the sequence begins 8, 10, 18, 28, 46, 74, 120, 194, 314, 508, …).

5) From the given information, $a=bu$, so plug this into the equation $a+b=cu$ to get $bu+b=cu$, and plug into the equation $b=(a-c)u$ to get $b=au-cu=bu^2-bu-b \Rightarrow 0=bu^2-bu-2b$ $=b(u^2-u-2)=b(u-2)(u+1)$. By assumption, $b\neq 0$, and if $u=-1$, then one of $a$ or $b$ would be negative, which violates our assumption. Therefore, $u=2$.

6) First, realize that $254016=504^2$, so $2^x=y^2-504^2=(y-504)(y+504)$. Each of these factors on the right must be powers of 2, and they differ by 1008. The only solution to $2^{m+1}+2^{m+2}+\ldots+2^{m+k}$ $=1008$ is found in the following way: $2^4\cdot 63=1008=\dfrac{2^{m+1}-2\cdot 2^{m+k}}{1-2}=2^{m+1}\left(2^k-1\right)\Rightarrow m=3$ and $k=6$, so the two factors are $2^4=16$ and $2^{10}=1024$. Therefore, $y=520$, making $2^x=16\cdot 1024$ $=2^{14}\Rightarrow x=14$. The ordered pair is $(14,520)$.

7) The average round trip speed is the harmonic mean of the two rates, so we are looking for the greatest integer $x<30$ such that $n=\dfrac{2\cdot 36\cdot x}{36+x}$ for some integer $n$. $36n+xn=72x\Rightarrow x=\dfrac{36n}{72-n}$. $n$ can't exceed 36, and the greatest $n$ yields the greatest $x$. A little trial and error reveals that working backwards from 36, the fastest average speed is $n=24$, which makes their return trip rate $x=18$.

8) $np+1$ is a multiple of $m$, $mp+1$ is a multiple of $n$, and $mn+1$ is a multiple of $p$. Therefore, $(np+1)(mp+1)(mn+1)=m^2n^2p^2+mnp^2+mn^2p+m^2np+np+mp+mn+1$ is a multiple of $mnp$, meaning that $np+mp+mn+1$ is a multiple of $mnp$, and $\dfrac{np+mp+mn+1}{mnp}=\dfrac{1}{m}+\dfrac{1}{n}+\dfrac{1}{p}+\dfrac{1}{mnp}\geq 1$ must be an integer. Further, $m$ can't be 4 or greater as the quantity would not be large enough, so consider casework on $m$:

a) If $m=1$, then $p+1$ is a multiple of $n$ and $n+1$ is a multiple of $p$. In this case, since $n<p$, $n+1\leq p\Rightarrow n+1=p$ (in order for $n+1$ to be a multiple of $p$), meaning that $n+2$ is a multiple of $n$, implying that $n=1$ (not possible since $m=1$) or $n=2\Rightarrow p=3$. So the only solution in this case is $(1,2,3)$.

b) If $m=2$, then $np+1$ is a multiple of 2, implying that $n$ and $p$ are both odd. $n$ would have to be less than 6 to make $\dfrac{1}{m}+\dfrac{1}{n}+\dfrac{1}{p}+\dfrac{1}{mnp}\geq 1$, so if $n=3$, some simple checking will reveal that $p=7$ makes this quantity equal to 1 ($p=5$ did not yield an integer for this quantity, and $p>7$ makes this quantity less than 1). So the only solution in this case is $(2,3,7)$.

c) If $m=3$, then then $np+1$ is a multiple of 3, implying that neither $n$ nor $p$ is a multiple of 3. Trying the least possible case of $(3,4,5)$, we see that $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{60}=\frac{48}{60}<1$, and increasing either or both of $n$ and $p$ will make this quantity decrease, there are no possible solutions in this case.

9) Let $a$, $b$, and $c$ be the three altitudes of a triangle, with $a\le b\le c$, and let $K$ be the area enclosed by the triangle. The three sides of the triangle must then be $\frac{2K}{a}$, $\frac{2K}{b}$, and $\frac{2K}{c}$, where $\frac{2K}{c}\le\frac{2K}{b}\le\frac{2K}{a}$. Therefore, by the triangle inequality, $2K\left(\frac{1}{c}+\frac{1}{b}\right)=\frac{2K}{c}+\frac{2K}{b}>\frac{2K}{a}$ $=2K\left(\frac{1}{a}\right)$, and since $K>0$, $\frac{1}{c}+\frac{1}{b}>\frac{1}{a}$. So for a given answer choice, if you look at the reciprocals of the numbers given, if the smaller two reciprocals have a sum greater than the third reciprocal, then those numbers can be the altitude lengths. For example, in letter choice A, $\frac{3}{2}+\frac{4}{1}=\frac{11}{2}>\frac{5}{1}$, so choice A can be the lengths of the three altitudes. Performing a similar calculation for the other choices, the answer choices that can be the lengths of the altitudes are A, B, C, E, F, and I; this makes the answer choices that aren't possible D, G, and H.

10) Let $z=\log_9 x=\log_{15} y=\log_{25}(x+y)$. Then $y^2=\left(15^z\right)^2=225^z=9^z25^z=x(x+y)\Rightarrow$ $0=x^2+xy-y^2$, and since $y\neq 0$, $0=\left(\frac{x}{y}\right)^2+\frac{x}{y}-1\Rightarrow\frac{x}{y}=\frac{-1\pm\sqrt{5}}{2}$, but since $x$ and $y$ are both positive, $\frac{x}{y}=\frac{-1+\sqrt{5}}{2}$.

Section V: Simple Graphs, Ridiculous Equations

1) $(x-3)^2+(y+4)^2=0\Rightarrow(x,y)=(3,-4)$

2) $4(x-3)^2+5(y+4)^2=0\Rightarrow(x,y)=(3,-4)$

3) $(2x+3y+2)(2x+3y-1)=0\Rightarrow y=-\frac{2}{3}x-\frac{2}{3}$ and $y=-\frac{2}{3}x+\frac{1}{3}$, so the slope is $m=-\frac{2}{3}$

4) $(2x-y-7)(x+y-8)=0\Rightarrow x+y=8$ and $2x-y=7\Rightarrow(x,y)=(5,3)$

5) $0=\left(x^2+y^2+2x-4y+5\right)\left(x^2+y^2-6x-14y+58\right)=\left((x+1)^2+(y-2)^2\right)\left((x-3)^2+(y-7)^2\right)$ $\Rightarrow(x,y)=(-1,2)$ or $(x,y)=(3,7)$

6) $0=\left(2x^2+y^2+12x-10y+43\right)\left(x^2+2y^2-10x+12y+43\right)=\left(2(x+3)^2+(y-5)^2\right)$ $\left((x-5)^2+2(y+3)^2\right)\Rightarrow(x,y)=(-3,5)$ or $(x,y)=(5,-3)$

7) $0=\left(x^2+y^2-2x+2y+2\right)\left(-3x+4y+6\right)=\left(\left(x-1\right)^2+\left(y+1\right)^2\right)\left(-3x+4y+6\right)$, so the point is $\left(1,-1\right)$ and the line is $-3x+4y+6=0$, so the distance from the point to the line is $\dfrac{\left|-3\left(1\right)+4\left(-1\right)+6\right|}{\sqrt{\left(-3\right)^2+4^2}}=\dfrac{1}{5}$.

8) $0=\left(3x+y-2\right)\left(-x+3y-6\right)\left(-2x-y+2\right)$, so the three lines are $3x+y-2=0$, $-x+3y-6=0$ and $-2x-y+2=0$, which all intersect at $\left(0,2\right)$.

9) $0=\left(x^2+y^2+4x-2y+5\right)\left(x^2+y^2-4x-2y+5\right)\left(x^2+y^2+8x+4y+20\right)=\left(\left(x+2\right)^2+\left(y-1\right)^2\right)$ $\left(\left(x-2\right)^2+\left(y-1\right)^2\right)\left(\left(x+4\right)^2+\left(y+2\right)^2\right)$, so the three points are $\left(-2,1\right)$, $\left(2,1\right)$, and $\left(-4,-2\right)$; the distance between the first two points is 4, and the third point is distance 3 away from the line $x=1$, so the area enclosed by the triangle is $\frac{1}{2}\left(4\right)\left(3\right)=6$.

10) $0=\left(x^2+y^2-6x+4y+13\right)^3=\left(\left(x-3\right)^2+\left(y+2\right)^2\right)^3\Rightarrow\left(x,y\right)=\left(3,-2\right)$

<u>Section VIII: You Must Be Joking, My Head Hurts Already</u>

1) see https://en.wikipedia.org/wiki/Mathematics_of_Sudoku
2) see https://en.wikipedia.org/wiki/Mathematics_of_Sudoku
3) see http://oeis.org/A001652
4) see http://www.math.wichita.edu/~richardson/heronian/heronian.html
5) By Descartes' Rule of Signs, this polynomial has exactly one negative root (and therefore one negative solution to the equation). The trick is recognizing this is close to a perfect cube and then factoring the result as a sum (or difference) of cubes:

$0=x^3+60x^2-150x+125=\left(-8x^3+60x^2-150x+125\right)+9x^3=\left(-2x+5\right)^3+9x^3=$ $\left(-2x+5+\sqrt[3]{9}x\right)\left(\left(-2x+5\right)^2-\left(-2x+5\right)\left(\sqrt[3]{9}x\right)+\left(\sqrt[3]{9}x\right)^2\right)$. Solving the first factor equal to 0 yields $5=2x-\sqrt[3]{9}x=\left(2-\sqrt[3]{9}\right)x\Rightarrow x=\dfrac{5}{2-\sqrt[3]{9}}$ (which is negative).

6) see http://oeis.org/A004215
7) Factored into linear and irreducible over integers quadratics,

$$y=\left(x+3\right)^2\left(x-3\right)\left(x+1\right)\left(x+4\right)\left(x-1\right)^2\left(x+2\right)\left(x^2-4x+5\right)\left(x^2-x+1\right)\left(x^2-5x+3\right)\left(x^2-4x+1\right)$$