For all questions, answer choice "E) NOTA" means none of the above answers is correct. Good luck…I'm sure you'll do *inte-great*!

1. What is the average value of $y = 4$ over the interval $[0, ¥)$?

A) 4 B) 0 C) ¥ D) 1 E) NOTA

2. An ox walks away from an infinitely small pulley pulling a rope slung over it. The rope is tied to the ox at a height 4 meters below the pulley. Suppose a barrel of calculus textbooks at the opposite end of the rope is rising at a constant 2 m/s. At what rate is the ox walking in m/s when it is 8 meters from being directly under the pulley?

A) $\sqrt{3}$ B) $\sqrt{5}$ C) $2\sqrt{2}$ D) 6 E) NOTA

3. Determine the absolute value of the difference between the absolute maximum and minimum of $f(x) = 3x - x^3$ on the interval $[0, 3]$.

A) 16 B) 18 C) 20 D) 22 E) NOTA

4. Find an initial value x_1 for the zero of $f(x) = 3x - x^3$ for which Newton's method gives an undefined quantity for x_2, the next iteration initial value.

A) -2 B) 0 C) 2 D) -18 E) NOTA

5. Cut four identical squares out of the corners of a 16 by 16 cm piece of cardboard and fold the sides to construct a box without a top. Find the area of the corner square cutout that maximizes the volume of the box.

A) $\frac{5}{3}$ B) $\frac{8}{3}$ C) $\frac{25}{9}$ D) $\frac{64}{9}$ E) NOTA

6. Stacy's mom is very concerned with her daughter Stacy's automotive fuel efficiency. The two of them make a trip to the beach one day, and Stacy's mom diligently records the car computer's digital read-outs of fuel consumption in gallons per hour every 5 min en route to the beach and produces the following chart:

Time (mins)	0	5	10	15	20	25	30	35	40	45	50	55	60
Gal/hr	2.5	2.4	2.3	2.2	2.4	2.5	2.6	2.5	2.4	2.3	2.4	2.4	2.3

Use trapezoidal rule with 4 equal subdivisions to approximate Stacy's total fuel consumption during the hour. If the car covered 57 mi during the 1-hour trip, using this approximation, what is Stacy's fuel efficiency in miles per gallon on the ride to the beach?

A) 21 B) 24 C) 42 D) 48 E) NOTA

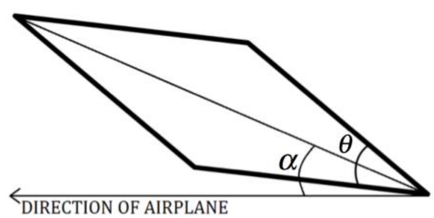
7. An Airbus A380-800 aircraft seats up to 853 passengers. Nolegroom Airlines can fill the first 400 seats of its A380-800 on route from Singapore to Salt Lake City at a fare of $1,000 per seat. For every $5 increase in the fare, the plane loses a passenger. For every decrease of $5, the flight gains a passenger. What fare (not necessarily *fair*) price per seat maximizes revenue?

A) $500 B) $800 C) $1200 D) $1500 E) NOTA

8. Evaluate $\int_{-6}^{2} \left|x^2+2x-8\right| dx$.

A) $\frac{64}{3}$ B) $\frac{98}{3}$ C) $\frac{100}{3}$ D) $\frac{152}{3}$ E) NOTA

9. Elon Musk, in his free time, is planning on founding another company, which will specialize in the manufacture of supersonic fighter jets. He drafts a design for an airplane wing that has a cross-section in the shape of a thin diamond (rhombus) in which the half-angle of opening is q and the attack angle is a. After doing some not-so-basic physics calculations along with folks from MIT, he finds that the ratio of lift to drag of the wing is given by the following equation: $\frac{lift}{drag} = \frac{a}{a^2+q^2}$.



For $q = \sqrt{p}/2$, find the best positive angle of attack a, which maximizes the lift to drag ratio.

A) $\frac{\sqrt{\pi}}{4}$ B) $\frac{1}{\sqrt{p}}$ C) $\frac{\sqrt{\pi}}{2}$ D) $\frac{p}{4}$ E) NOTA

10. (Continued from #9...) For $q = \sqrt{p}/2$, find the minimum lift to drag ratio among all negative angles a. (This attack angle could be used in the design of a winged car attempting to break the sound barrier to prevent it from flying.)

A) -1 B) $-\frac{\sqrt{\pi}}{2}$ C) $-\frac{1}{\sqrt{p}}$ D) $-\frac{1}{2\sqrt{p}}$ E) NOTA

11. As a part of training for MAΘ nationals, your math club decides to hit the weight room. Dr. Morris, who happens to be in the same weight room, lifts a bag of sand, originally weighing 1200N (he's a pretty strong guy), at a constant rate. As it rises, sand leaks out at a constant rate. The sand is half gone by the time the bag has been lifted 4m. How much work was done in lifting the bag 4m? (Neglect the weights of the bag and lifting equipment; Hint: $W = ò F(x)dx$)

A) 2400J B) 3600J C) 4200J D) 4800J E) NOTA

12. The cycloid given parametrically by $x = t - \sin t,\ y = 1 - \cos t$ describes the path of a point travelling along a wheel rolling at a constant speed, where t represents time. How fast is the point moving at each time t?

A) $\sqrt{2 - 2\cos t}$ B) $\sqrt{2 + 2\cos t}$ C) $\sqrt{2 - 2\sin t}$ D) $\sqrt{2 + 2\sin t}$ E) NOTA

13. (Continued from #12…) Find the length of the cycloid for one half turn of the wheel.

A) $2\sqrt{2}$ B) 4 C) $4\sqrt{2}$ D) 8 E) NOTA

14. Find a value t for which the solution $y = f(t)$ of the differential equation $y' = y^2 + 1$ given the initial condition $y(0) = 1$ is undefined.

A) $-\frac{3\pi}{4}$ B) $-\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π E) NOTA

15. Determine the slope of the tangent line to $r = 2\sin(2\theta)$ at $\theta = \frac{\pi}{6}$.

A) $\frac{\sqrt{3}}{9}$ B) $\frac{2\sqrt{3}}{9}$ C) $\frac{2\sqrt{3}}{3}$ D) $\frac{5\sqrt{3}}{3}$ E) NOTA

16. Let R be the region in the 1st quadrant of the (x,y) plane $(x \geq 0, y \geq 0)$ trapped under the first hump (the one between the least x-values) of $y = \sin(x^2)$. Calculate the volume of the solid produced by revolving R around the y-axis.

A) 1 B) 2π C) $2\sqrt{2}\pi$ D) 4π E) NOTA

17. Calculate the derivative of the function $y = x^x \ln x$ evaluated at $x = e$.

A) $e^e(1+2e)$ B) $e^{e-1}(2+e)$ C) $e^e(2+e)$ D) $e^{e-1}(1+2e)$ E) NOTA

18. Find the volume of the solid produced by revolving the ellipse $\frac{(x-5)^2}{4} + \frac{(y-4)^2}{9} = 1$ about the line $y + x = 1$.

A) $24\sqrt{2}\pi^2$ B) $48\sqrt{2}\pi^2$ C) $96\pi^2$ D) $192\pi^2$ E) NOTA

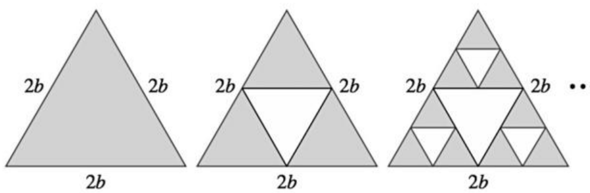
19. Consider the cone generated by rotating the line segment $y = mx$, $0 \le x \le r$, $y \ge 0$ around the y-axis. Find the slope $m > 0$ the will maximize the volume, assuming that its lateral surface area is fixed.

A) $\frac{\sqrt{2}}{2}$ B) 1 C) $\sqrt{3}$ D) $2\sqrt{2}$ E) NOTA

20. In many chemical reactions, the rate at which the amount of a substrate changes is proportional to the amount present. For example, the differential equation for the change of δ-gluconolactone into glutonic acid is $\frac{dy}{dt} = -0.6y$, where t is measured in hours. If there are 100 grams of δ-gluconolactone present when $t = 0$, how many grams will be left after the first 100 minutes of the reaction?

A) $\frac{100}{e^{60}}$ B) $\frac{100}{e}$ C) $\frac{100}{e^{.6}}$ D) 40 E) NOTA

21. Fractals can be pretty cool. A simple fractal can be constructed by removing "upside down" equilateral triangles with vertices at side midpoints from a single large "right side up" equilateral triangle with side length $2b$ in the manner depicted below, indefinitely. The sum of the areas removed from the original triangle forms an infinite series. Calculate the sum of this infinite series and hence the total area removed from the original triangle.



A) $b^2\sqrt{2}$ B) $b\sqrt{3}$ C) $b^2\sqrt{3}$ D) $4b^2\sqrt{3}$ E) NOTA

22. Simplify $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n+2}$

A) e^2 B) $e^2 + 2e$ C) $2e^2$ D) e^4 E) NOTA

23. What is the rate of change of $\sqrt{1+x^2}$ with respect to $\frac{x^2}{\sqrt{1+x^2}}$ evaluated at $x = 0$?

A) 0 B) $\frac{1}{2}$ C) $\frac{2}{3}$ D) $\frac{\sqrt{2}}{2}$ E) NOTA

24. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{e}{i+n}$ simplifies to the form $e\ln b$. Compute $\lceil e \rceil + \lfloor b \rfloor$.

A) 3 B) 4 C) 5 D) 6 E) NOTA

25. Nate's favorite function, $y = N(x)$, is positive and continuous. A solid is generated by revolving about the x-axis the region bounded by the graph of $N(x)$, the x-axis, the fixed line $x = a$, and the variable line $x = b, b > a$. Its volume, for all b, is $\pi(b^2 - ab)$. Determine $N\left(\frac{a+1}{2}\right)$.

A) $\sqrt{a}$ B) $\sqrt{2}$ C) $\frac{1}{\sqrt{\pi}}$ D) $\frac{\sqrt{a}}{2}$ E) NOTA

And now for a short segment on my personal favorite mathematician and the GOAT, Leonhard Euler…

26. We know that Euler's Method is a useful numerical technique for approximating solutions to initial value problems $y' = f(x, y)$, $y(x_0) = y_0$ that cannot be solved through conventional means. It follows the general format of $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})dx$, where $x_n = x_{n-1} + dx$ and dx represents the step size for each iteration.

We can improve on Euler's Method by taking an average of two slopes. In Improved Euler's Method, we first estimate y_n like usual, but denote it by z_n. We then take the average of $f(x_{n-1}, y_{n-1})$ and $f(x_n, z_n)$ in place of $f(x_{n-1}, y_{n-1})$ in the next step. Thus, Improved Euler's Method is

$$z_n = y_{n-1} + f(x_{n-1}, y_{n-1})dx$$

$$y_n = y_{n-1} + \left(\frac{f(x_{n-1}, y_{n-1}) + f(x_n, z_n)}{2}\right)dx$$

Using Improved Euler's Method, approximate $y(1)$ for $y' = 2y(x+1)$ given $y(0) = 3$ with step size $dx = .5$.

A) 38.25 B) 45.75 C) 59.25 D) 66.75 E) NOTA

27. Euler's gamma function $\Gamma(x)$ is significant in computing probabilities using all positive complex numbers. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,\ x > 0$$

Use integration by parts to determine an expression equivalent to $\Gamma(x+1)$.

A) $(x-1)!$ B) $(x+1)!$ C) ${}_{(x+1)}C_{(x-1)}$ D) ${}_{(x+1)}P_{(x-1)}$ E) NOTA

28. Find a particular value of a , the following limit is finite. Evaluate the limit for this value of a:

$$\lim_{x\to 0}\frac{\sin(ax)-\sin x - x}{x^3}$$

A) $-\frac{7}{6}$ B) $-\frac{1}{6}$ C) $\frac{1}{6}$ D) $\frac{5}{6}$ E) NOTA

29. Find the sum of the infinite series:

$$1+\frac{1}{2}\bullet 2+\frac{1}{4}\bullet 3+\cdots+\frac{n}{2^{n-1}}+\cdots$$

(*Hint*: differentiate both sides of the equation $1/(1-x)=1+\sum_{n=1}^{\infty}x^n$)

A) $3\sqrt{2}$ B) $2\sqrt{5}$ C) $\frac{\pi^2}{2}$ D) 2π E) NOTA

30. Everyone loves a good ol' math proof, assuming it's correct, of course. Your Friendly Neighborhood Mathemagician tries to convince you that $\ln 3$ is equal to $\infty-\infty$. In what line does the first error occur?

$$1. \ln 3=\ln 1+\ln 3=\ln 1-\ln\left(\frac{1}{3}\right)$$

$$2. =\lim_{b\to\infty}\ln\left(\frac{b-2}{b}\right)-\ln\left(\frac{1}{3}\right)$$

$$3. =\lim_{b\to\infty}\left[\ln\left(\frac{x-2}{x}\right)\right]_3^b$$

$$4. =\lim_{b\to\infty}\left[\ln(x-2)-\ln x\right]_3^b$$

$$5. =\lim_{b\to\infty}\int_3^b\left(\frac{1}{x-2}-\frac{1}{x}\right)dx$$

$$6. =\int_3^\infty\left(\frac{1}{x-2}-\frac{1}{x}\right)dx$$

$$7. =\int_3^\infty\frac{1}{x-2}dx-\int_3^\infty\frac{1}{x}dx$$

$$8. =\lim_{b\to\infty}\left[\ln(x-2)\right]_3^b-\lim_{b\to\infty}\left[\ln x\right]_3^b$$

$$9. =\infty-\infty$$

A) 3 B) 5 C) 7 D) 8 E) NOTA