Answers:

1. B
2. C
3. B
4. A
5. C
6. D
7. A
8. B
9. C
10. D
11. B
12. B
13. A
14. D
15. A
16. D
17. A
18. C
19. B
20. C
21. A
22. D
23. D
24. C
25. B
26. C
27. A
28. D
29. E
30. A

Solutions:

1. By the Law of Cosines, $\cos\alpha = \frac{4^2+7^2-9^2}{2\cdot 4\cdot 7} = -\frac{2}{7}$, where α is the obtuse angle of the parallelogram. Therefore, $\cos\left(180^\circ - \alpha\right) = \frac{2}{7}$, and using the Law of Cosines again, where x is the length of the shorter diagonal, $x^2 = 4^2+7^2-2\cdot 4\cdot 7\cdot \frac{2}{7} = 49 \Rightarrow x = 7$.

2. $38 = 2+\left(75-3\right)d \Rightarrow d = 1/2$. Therefore, $a_{2015} = 2+\left(2015-3\right)\cdot 1/2 = 1008$

3. $2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0 \Rightarrow 6\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} + 2\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} + 3 = 0 \Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = -\frac{3}{8}$

4. $\frac{\binom{2}{1}\binom{26}{2}+\binom{13}{1}\binom{4}{2}-2\cdot\binom{2}{1}\binom{13}{1}\binom{2}{2}}{\binom{52}{2}} = \frac{2\cdot 325+13\cdot 6-2\cdot 2\cdot 13\cdot 1}{1326} = \frac{676}{1326} = \frac{26}{51}$

5. This is the formula for the number of derangements of *n* objects, and by plugging in the numbers, one gets $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, and $D_6 = 265$.

6. Since $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, $\left.\frac{dy}{dx}\right|_{x=0} = 1$, so the tangent line approximation is $L\left(x\right) = 0+1\left(x-0\right) = x$. Therefore, for *x*-values close to 0, $\sin^{-1}x \approx x$. Thus, $\sin^{-1}\left(0.2\right) \approx 0.2$.

7. Since $\lim_{x\to-\infty}\left(8-5x+x^3\right) = -\infty$ (based on the graph of $y = 8-5x+x^3$), $\lim_{x\to-\infty}\left(e^{8-5x+x^3}\right) = 0$. Additionally, since $\lim_{x\to-\infty}\left(\frac{2x-6x^2}{4+x+3x^2}\right) = -2$ (the ratio of leading coefficients), $\lim_{x\to-\infty}\left(e^{\frac{2x-6x^2}{4+x+3x^2}}\right) = e^{-2}$. Therefore, $\lim_{x\to-\infty}\left(e^{8-5x+x^3}+e^{\frac{2x-6x^2}{4+x+3x^2}}\right) = 0+e^{-2} = e^{-2}$.

8. $\cosh x = \frac{d}{dx}\left(\sinh x\right) = \frac{e^x+e^{-x}}{2}$, so $\cosh x - \sinh x = \frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2} = e^{-x}$

9. $f'(x)=12x^3+24x^2-60x-72=12(x-2)(x+1)(x+3)$, so the only critical number in the given interval is -1. Since $f(-1)=61, f(1)=-67$, and $f(-2)=32$, the maximum value is 61, the minimum value is -67, and the positive difference in these values is $61-(-67)=128$.

10. Rewriting this equation in standard form yields $\frac{(y+1)^2}{16}-\frac{(x-4)^2}{12}=1$, so the length of the latus rectum is $\frac{2b^2}{a}=\frac{2\cdot 12}{4}=6$.

11. $S=\sum_{n=1}^{\infty}\left((2n^2+n+1)\left(3/4\right)^n\right)=4\left(3/4\right)+11\left(3/4\right)^2+22\left(3/4\right)^3+37\left(3/4\right)^4+\ldots$. Multiplying this by $3/4$ yields $3/4\,S=4\left(3/4\right)^2+11\left(3/4\right)^3+22\left(3/4\right)^4+\ldots$, and subtracting the second equation from the first yields $1/4\,S=4\left(3/4\right)+7\left(3/4\right)^2+11\left(3/4\right)^3+15\left(3/4\right)^4+\ldots$. Multiply this new equation by $3/4$ to get $3/16\,S=4\left(3/4\right)^2+7\left(3/4\right)^3+11\left(3/4\right)^4+\ldots$, then subtract from the previous equation to get $1/16\,S=4\left(3/4\right)+3\left(3/4\right)^2+4\left(3/4\right)^3+4\left(3/4\right)^4+\ldots$. On the right hand side of this equation, beginning with the third term, the series is geometric, so this equation becomes $1/16\,S=4\left(3/4\right)+3\left(3/4\right)^2+\frac{4\left(3/4\right)^3}{1-3/4}=3+\frac{27}{16}+\frac{27}{4}=\frac{183}{16}\Rightarrow S=183$.

12. Let (x,y) be a point on the parabola. The distance to the point $(0,2)$ is defined as $d=\sqrt{x^2+(y-2)^2}=\sqrt{(y-1)+(y-2)^2}=\sqrt{y^2-3y+3}$, and since the quantity under the radical is positive, we need only minimize that quantity. $f(y)=y^2-3y+3$ is quadratic, so the minimum value is when $y=-\frac{-3}{2\cdot 1}=\frac{3}{2}$. Plugging this back in yields $x^2=\frac{1}{2}\Rightarrow x=\pm\frac{\sqrt{2}}{2}$. Since x has to be non-negative, the closest point is $\left(\frac{\sqrt{2}}{2},\frac{3}{2}\right)$.

13. The number of subsets of $\{x\in\mathbb{Z}\,|\,3\le x\le 10\}$ is $2^8=256$. For the set $\{1,2\}$, there are four subsets: $\varnothing$, $\{1\}$, $\{2\}$, and $\{1,2\}$. The union of the second or third of these subsets with each of the 256 subsets of $\{x\in\mathbb{Z}\,|\,3\le x\le 10\}$ is the list of all such subsets, which is $2\cdot 256=512$.

14. $\ln y = (\ln x)^2 \Rightarrow \dfrac{\frac{dy}{dx}}{y} = \dfrac{2\ln x}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{2y\ln x}{x} = \dfrac{2x^{\ln x}\ln x}{x}$. Therefore, $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{\frac{2x^{\ln x}\ln x}{x}}{2^{\ln x}\ln 2\cdot\frac{1}{x}}$

$= \dfrac{2x^{\ln x}\ln x}{2^{\ln x}\ln 2} \Rightarrow \left.\dfrac{dy}{dz}\right|_{x=2} = \dfrac{2\cdot 2^{\ln 2}\ln 2}{2^{\ln 2}\ln 2} = 2$.

15. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{i}{i^2+n^2} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{1}{n}\cdot\dfrac{\frac{i}{n}}{\frac{i^2}{n^2}+1} = \int_0^1 \dfrac{x}{x^2+1}dx = \dfrac{1}{2}\ln\left(x^2+1\right)\Big|_0^1 = \dfrac{1}{2}\ln 2 - \dfrac{1}{2}\ln 1 = \dfrac{1}{2}\ln 2$

16. $v = \int a\,dt = \dfrac{1}{2}t^2 - 2t + c \Rightarrow 3 = \dfrac{1}{2}\cdot 2^2 - 2\cdot 2 + c \Rightarrow c = 5 \Rightarrow v = \dfrac{1}{2}t^2 - 2t + 5$. Therefore, the displacement is $\int_1^3\left(\dfrac{1}{2}t^2 - 2t + 5\right)dt = \left(\dfrac{1}{6}t^3 - t^2 + 5t\right)\Big|_1^3 = \left(\dfrac{27}{6} - 9 + 15\right) - \left(\dfrac{1}{6} - 1 + 5\right) = \dfrac{19}{3}$.

17. Let $y = mx + b$, where m and b are constants and $m \neq 0$. Plugging this into the differential equation in (A) yields the equation $m = 2x + mx + b$, so make $m = b = -2$, and there are no restrictions on x or y. To see why this won't work for the other answer choices, use the same argument. For (B), $m = 2x(mx+b)$, and the only ways to get this to work are if $m = b = 0$, which is invalid, or if x is a specific number, which is also invalid. For (C), $m = \dfrac{2x}{mx+b}$, which does have the solution $b = 0$ and $m = \pm\sqrt{2}$; however, this solution cannot contain the origin and is therefore not a line. For (D), the equation becomes $0 = 1$, an impossibility.

18. The integrand, along with the constant multiple, is the standard normal curve function, so because the integral's limits are -1 and 1, the integral is the area within one standard deviation of the mean—in other words, 0.68.

19. Let x and y be the two legs of the right triangle, and let P be the triangle's perimeter. Since twenty minutes elapse, the legs at that moment have lengths 10 and 20 miles. Since

$P = x + y + \sqrt{x^2+y^2}$, $\dfrac{dP}{dt} = \dfrac{dx}{dt} + \dfrac{dy}{dt} + \dfrac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}} \Rightarrow \dfrac{dP}{dt} = 30 + 60 + \dfrac{2\cdot 10\cdot 30 + 2\cdot 20\cdot 60}{2\sqrt{10^2+20^2}}$

$= 90 + \dfrac{3000}{20\sqrt{5}} = 90 + 30\sqrt{5}$.

20. The two curves intersect at the points $(0,0)$, $(1,2)$, and $(2,4)$, and based on the graphs of both, the area is $\int_0^1\left(x^3-3x^2+4x-2x\right)dx+\int_1^2\left(2x-\left(x^3-3x^2+4x\right)\right)dx=\int_0^1\left(x^3-3x^2+2x\right)dx$ $+\int_1^2\left(-x^3+3x^2-2x\right)dx=\left(\frac{x^4}{4}-x^3+x^2\right)\Big|_0^1+\left(-\frac{x^4}{4}+x^3-x^2\right)\Big|_1^2=\frac{1}{4}-1+1-0+0-0-4+8-4+\frac{1}{4}$ $-1+1=\frac{1}{2}$.

21. First, $2r\frac{dr}{d\theta}=-6\sin(3\theta)\Rightarrow\frac{dr}{d\theta}=-\frac{3\sin(3\theta)}{r}$. Then, since $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}$

$$=\frac{r\cos\theta+\frac{dr}{d\theta}\sin\theta}{-r\sin\theta+\frac{dr}{d\theta}\cos\theta}=\frac{r\cos\theta-\frac{3\sin(3\theta)}{r}\sin\theta}{-r\sin\theta-\frac{3\sin(3\theta)}{r}\cos\theta}=\frac{r^2\cos\theta-3\sin(3\theta)\sin\theta}{-r^2\sin\theta-3\sin(3\theta)\cos\theta}$$

$$=\frac{2\cos(3\theta)\cos\theta-3\sin(3\theta)\sin\theta}{-2\cos(3\theta)\sin\theta-3\sin(3\theta)\cos\theta},\ \frac{dy}{dx}\Big|_{\theta=\frac{2\pi}{3}}=\frac{2\cdot1\cdot-\frac{1}{2}-3\cdot0\cdot\frac{\sqrt{3}}{2}}{-2\cdot1\cdot\frac{\sqrt{3}}{2}-3\cdot0\cdot-\frac{1}{2}}=\frac{-1}{-\sqrt{3}}=\frac{\sqrt{3}}{3}.$$

22. The two curves intersect at the points $(1,1)$ and $(2,2)$, and using the shells method, because the region is revolved about $x=-1$, the radius is $x+1$. This makes the volume $2\pi\int_1^2(x+1)\left(\left(-x^2+4x-2\right)-\left(x^2-2x+2\right)\right)dx=2\pi\int_1^2(x+1)\left(-2x^2+6x-4\right)dx$ $=2\pi\int_1^2\left(-2x^3+4x^2+2x-4\right)dx=2\pi\left(-\frac{x^4}{2}+\frac{4x^3}{3}+x^2-4x\right)\Big|_1^2=2\pi\left(\left(-8+\frac{32}{3}+4-8\right)-\left(-\frac{1}{2}+\frac{4}{3}\right.\right.$ $\left.\left.+1-4\right)\right)=\frac{5\pi}{3}$.

23. $\frac{dy}{dx}=\frac{-\sin x}{\cos x}=-\tan x\Rightarrow\sqrt{1+(-\tan x)^2}=\sqrt{1+\tan^2 x}=\sqrt{\sec^2 x}=|\sec x|$. Also, since the length is for $0\le x\le{}^{\pi}\!/_4$, $|\sec x|=\sec x$; therefore, the length is $\int_0^{\pi/4}\sec x\,dx=\ln|\sec x+\tan x|\Big|_0^{\pi/4}$ $=\ln\left(\sqrt{2}+1\right)-\ln(1+0)=\ln\left(\sqrt{2}+1\right)$.

24. The average value is $\frac{1}{e-1}\int_1^e\left(3x^2-2x+1\right)dx=\frac{1}{e-1}\left(x^3-x^2+x\right)\Big|_1^e=\frac{1}{e-1}\left(\left(e^3-e^2+e\right)\right.$

$-\left(1-1+1\right)\right)=\dfrac{e^3-e^2+e-1}{e-1}=\dfrac{\left(e^2+1\right)\left(e-1\right)}{e-1}=e^2+1$.

25. The differential equation is first-order because no derivative higher than the first is present. It is not homogeneous or autonomous because of the $3x$ term. It is ordinary because it features y as a function of x only. It is not separable because $3x+2y$ cannot be written as a function of *x* times a function of *y*. It is not linear because the sum of two solutions is not also a solution (since you would need two $3x$ terms, one for each solution). Therefore, only adjectives I and III apply.

26. $\int_{1.5}^{2}\dfrac{1}{\sqrt{2x-x^2}}dx=\lim_{n\to 2^-}\int_{1.5}^{n}\dfrac{1}{\sqrt{1-\left(x-1\right)^2}}dx=\lim_{n\to 2^-}\left(\sin^{-1}\left(x-1\right)\right)\Big|_{1.5}^{n}=\lim_{n\to 2^-}\left(\sin^{-1}\left(n-1\right)-\sin^{-1}0.5\right)$

$=\sin^{-1}1-\sin^{-1}0.5=\pi/2-\pi/6=\pi/3$.

27. $\int_{1}^{\infty}\dfrac{\sqrt{x}}{1+x^3}dx=\lim_{n\to\infty}\int_{1}^{n}\dfrac{\sqrt{x}}{1+x^3}dx$. Now make the substitutions $u=x^{3/2}$, $du=\dfrac{3}{2}x^{1/2}dx$ $\left(\dfrac{2}{3}du=x^{1/2}dx\right)$ to get $\dfrac{2}{3}\lim_{n\to\infty}\int_{1}^{n^{3/2}}\dfrac{1}{1+u^2}du=\dfrac{2}{3}\lim_{n\to\infty}\left(\tan^{-1}u\right)\Big|_{1}^{n^{3/2}}=\dfrac{2}{3}\lim_{n\to\infty}\left(\tan^{-1}\left(n^{3/2}\right)-\tan^{-1}1\right)$

$=\dfrac{2}{3}\left(\dfrac{\pi}{2}-\dfrac{\pi}{4}\right)=\dfrac{2}{3}\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{6}$.

28. $\sum_{i=2}^{\infty}\left(\dfrac{2^i+3^i}{5^i}\right)=\sum_{i=2}^{\infty}\left(\dfrac{2}{5}\right)^i+\sum_{i=2}^{\infty}\left(\dfrac{3}{5}\right)^i=\dfrac{4/25}{1-2/5}+\dfrac{9/25}{1-3/5}=\dfrac{4/25}{3/5}+\dfrac{9/25}{2/5}=\dfrac{4}{15}+\dfrac{9}{10}=\dfrac{7}{6}$

29. $\lim_{n\to\infty}\left|\dfrac{\sqrt{n+1}}{1+\left(n+1\right)^3}\cdot\dfrac{1+n^3}{\sqrt{n}}\right|=\lim_{n\to\infty}\sqrt{\dfrac{n+1}{n}}\cdot\lim_{n\to\infty}\dfrac{1+n^3}{1+\left(n+1\right)^3}=1\cdot 1=1$, so the Ratio Test cannot be used to conclude convergence of the series A (though comparison to a power series can).

$\lim_{n\to\infty}\left|\dfrac{\left(-1\right)^{n+2}\left(n+1\right)}{\left(n+1\right)^2+1}\cdot\dfrac{n^2+1}{\left(-1\right)^{n+1}n}\right|=\lim_{n\to\infty}\dfrac{n^3+n^2+n+1}{n^3+2n^2+2n}=1$, so the Ratio Test cannot be used to conclude convergence of the series B (though Alternating Series test can).

$\lim_{n\to\infty}\left|\dfrac{2}{\left(n+2\right)\left(n+4\right)}\cdot\dfrac{\left(n+1\right)\left(n+3\right)}{2}\right|=\lim_{n\to\infty}\dfrac{n^2+4n+3}{n^2+6n+8}=1$, so the Ratio Test cannot be used to conclude convergence of the series C (though it is a telescoping series).

$$\lim_{n\to\infty}\left|\frac{1}{(n+1)\ln(n+1)^2}\cdot\frac{n(\ln n)^2}{1}\right|=\lim_{n\to\infty}\frac{n}{n+1}\cdot\left(\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right)^2=1\cdot\left(\lim_{n\to\infty}\frac{1/n}{1/(n+1)}\right)^2=1\cdot 1^2=1$$, so the Ratio Test cannot be used to conclude convergence of the series D (though the Integral test can).

Therefore, none of the convergent series can be shown to converge by the Ratio Test.

30. $1+2+\ldots+x=(x+1)+(x+2)+\ldots.+(x+p)\Rightarrow\frac{x(x+1)}{2}=\frac{p}{2}\left((x+1)+(x+p)\right)$

$$x^2+x=2xp+p+p^2\Rightarrow 0=p^2+(2x+1)p-\left(x^2+x\right)\Rightarrow p=\frac{-(2x+1)\pm\sqrt{4x^2+4x+1+4\left(x^2+x\right)}}{2\cdot 1}$$

$\Rightarrow p=\frac{-2x-1+\sqrt{8x^2+8x+1}}{2}$ $(\text{since p}>0)$, so we are looking for the least $x>84$ such that $8x^2+8x+1=2(2x+1)^2-1$ is a perfect square. Setting this perfect square as k^2 and setting $q=2n+1$, we are trying to solve the Pell Equation $k^2-2q^2=-1$. In trying to solve $k^2-2q^2=\pm1$, find the smallest solution in positive integers, which is $k=q=1$. All solutions are found recursively in the following way:

1. The next q is the sum of the previous k and q.
2. The next k is the sum of the previous k and twice the previous q.

The table looks like this:

k	q	k^2-2q^2	x (only when q is odd)
1	1	-1	0
3	2	1	
7	5	-1	2
17	12	1	
41	29	-1	14
99	70	1	
239	169	-1	84
577	408	1	
1393	985	-1	492

It can be shown also that the 1 and -1 will always alternate, so the next value of x will be 492. In fact, were the last column to be continued, the numbers in the last column would be the only numbers that have the property defined in this question (with the exception of 0, of course).