Helpful Hints

This test contains several tricky integrals. Here are some techniques that may be helpful in solving them.

- **Improper integrals:** Treat $\int_a^\infty f(x)dx$ as $\lim_{c\to\infty}\int_a^c f(x)dx$. If $f(x)$ is discontinuous at $x=b$, treat $\int_a^b f(x)dx$ as $\lim_{c\to b}\int_a^c f(x)dx$.
- **Integration by parts:** $\int_a^b u(x)\frac{dv}{dx}dx = [u(x)v(x)]_a^b - \int_a^b v(x)\frac{du}{dx}dx$.
- **Trigonometric substitution:** Letting $x = \sin(\theta)$, $\sec(\theta)$, or $\tan(\theta)$ can be helpful with expressions such as $x^2 \pm a^2$ or $a^2 - x^2$.
- The so-called **Weierstrass substitution:** For hard trigonometric integrals, letting $t=\tan\left(\frac{x}{2}\right)$ so that $\sin(x) = \frac{2t}{1+t^2}$, $\cos(x) = \frac{1-t^2}{1+t^2}$, and $dx = \frac{2}{1+t^2}dt$, can be helpful.
- Feynman's **Differentiation under the integral sign:** Under the right circumstances, if $I(y) = \int f(x,y)dx$, then differentiating so that $I'(y) = \int \frac{d}{dy}[f(x,y)]dx$ can make an unsolvable integral solvable.
- **Unnamed trick #1:** $\int\left(e^x f(x) + e^x f'(x)\right)dx = \int \frac{d}{dx}\left(e^x f(x)\right)dx = e^x f(x) + c$.
- **Unnamed trick #2:** $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

For all questions, NOTA means None Of These Answers.

(1) $\int_0^1 e^2 dx =$

(a) $\frac{1}{3}$ (b) $e^2 - 1$

(c) e^2 (d) $\frac{1}{2}$ (e) NOTA

(2) $\int_0^1 (2x^3 - x + 1)dx =$

(a) 1 (b) -1

(c) 2 (d) $\frac{3}{2}$ (e) NOTA

(3) $\int_0^{\frac{\pi}{2}} \cos(3x)\,dx =$

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$

(c) 1 (d) -1 (e) NOTA

(4) $\int_{-1}^{1} |x| dx =$

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) 2 (e) NOTA

(5) $\int_{1}^{5} \frac{1}{x-2} dx =$

(a) $\ln(3)$ (b) $\ln(2)$

(c) 0 (d) $\frac{8}{9}$ (e) NOTA

(6) $\int_{0}^{\sqrt{\ln(2)}} xe^{x^2} dx =$

(a) 1 (b) $e-1$

(c) $\frac{1}{2}(e-1)$ (d) $\frac{1}{2}$ (e) NOTA

(7) $\int_{0}^{1} \frac{x-1}{x^2-2x-3} dx =$

(a) $\ln\left(\frac{2\sqrt{3}}{3}\right)$ (b) $\ln(2\sqrt{3})$

(c) $\ln\left(\frac{\sqrt{3}}{2}\right)$ (d) $\ln\left(\frac{4}{3}\right)$ (e) NOTA

(8) $\int_{0}^{1} \frac{1}{x^2-2x-3} dx =$

(a) $\frac{\ln(\sqrt{3})}{2}$ (b) $\frac{\ln(3)}{4} - 1$

(c) $-\frac{\ln(3)}{4}$ (d) $\frac{\ln(3)}{4}$ (e) NOTA

(9) $\int_{-1}^{1} \frac{1}{x^2+2x+5} dx =$

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$ (e) NOTA

(10) $\int_{0}^{1} \sqrt{1-\sqrt{x}} dx =$

(a) $\frac{1}{3}$ (b) $\frac{8}{15}$

(c) $\frac{2}{3}$ (d) $\frac{4}{15}$ (e) NOTA

(11) $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \sin\left(\frac{x-x^3}{2}\right)\cos\left(\frac{x+x^3}{2}\right)dx =$

(a) $\frac{\pi^4}{256}$ (b) $\frac{\pi^2}{16}$

(c) $\frac{\pi}{8}$ (d) 0 (e) NOTA

(12) $\int_0^{\frac{\pi}{12}} \sec(x)\,dx =$

(a) $\ln\left(4-\sqrt{2}+\sqrt{6}\right)-\ln\left(\sqrt{2}+\sqrt{6}\right)$ (b) $\ln\left(4+\sqrt{2}+\sqrt{6}\right)-\ln\left(\sqrt{2}-\sqrt{6}\right)$

(c) $\ln\left(4-\sqrt{2}+\sqrt{6}\right)-\ln\left(\sqrt{6}-\sqrt{2}\right)$ (d) $\ln\left(4+\sqrt{2}+\sqrt{6}\right)-\ln\left(\sqrt{6}-\sqrt{2}\right)$

(e) NOTA

(13) $\int_0^2 \sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}\,dx =$

(a) $\frac{5}{6}$ (b) $\frac{13}{6}$

(c) $\frac{19}{6}$ (d) $-\frac{7}{6}$ (e) NOTA

(14) $\int_0^1 x^2 e^{2x} dx =$

(a) $\frac{e^2-1}{2}$ (b) e^2-1

(c) $\frac{e^2-1}{4}$ (d) $\frac{e^2}{4}-1$ (e) NOTA

(15) Find $\int_0^{\infty} e^{-\alpha x}\sin(x)\,dx$ for real $\alpha > 0$.

(a) $-\frac{1}{\alpha^2-1}$ (b) $\frac{1}{\alpha^2-1}$

(c) $-\frac{1}{\alpha^2+1}$ (d) $\frac{1}{\alpha^2+1}$ (e) NOTA

(16) $\int_0^2 \sqrt{4-x^2}\,dx =$

(a) 4π (b) 2π

(c) π (d) $\frac{\pi}{2}$ (e) NOTA

(17) $\int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x}\,dx =$

(a) $\frac{\sqrt{2}}{2}$ (b) $1-\frac{\pi}{4}$

(c) $1+\pi$ (d) $\frac{\pi}{2}$ (e) NOTA

(18) $\int_1^{\ln(2)} \ln\left(x^{e^x} \cdot e^{\left(\frac{e^x}{x}\right)}\right) dx =$

(a) 2 (b) $2\ln(ln(2))$

(c) $\ln(4)$ (d) $e^2\ln(2)$ (e) NOTA

(19) $\int_0^{\pi} \frac{1}{3+\cos(x)}\,dx =$

(a) $\frac{\sqrt{2}}{4}\pi$ (b) $\frac{\sqrt{3}}{3}\pi$

(c) $\frac{\sqrt{2}}{2}\pi$ (d) $\frac{\sqrt{3}}{2}\pi$ (e) NOTA

(20) $\int_0^1 \frac{1}{1+\left(1-\frac{1}{x}\right)^{2015}}\,dx =$

(a) 2015 (b) $\frac{1}{2}$

(c) $\arctan(2015)$ (d) $\frac{1}{2015}$ (e) NOTA

(21) Solve for a: $\int_1^a \frac{6}{x^4}\,dx = 1$

(a) 2 (b) $\sqrt[3]{2}$

(c) 1.5 (d) $\sqrt{2}$ (e) NOTA

(22) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(n+k)^2} =$

(a) ∞ (b) $e-\frac{1}{2}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$ (e) NOTA

For Problems (23) to (26), let R be the region bounded by the functions $f(x) = -x(x-3)$ and $g(x) = x$.

(23) Find the area of region R.

(a) $\frac{4}{3}$ (b) $-\frac{4}{3}$

(c) $\frac{8}{3}$ (d) $-\frac{8}{3}$ (e) NOTA

(24) Find the volume when region R is rotated about the x-axis.

(a) $\frac{61}{15}\pi$ (b) $\frac{46}{15}\pi$

(c) $\frac{71}{15}\pi$ (d) $\frac{56}{15}\pi$ (e) NOTA

(25) Find the volume when region R is rotated about the y-axis.

(a) $\frac{4}{3}\pi$ (b) $\frac{16}{15}\pi$

(c) $\frac{8}{3}\pi$ (d) $\frac{32}{15}\pi$ (e) NOTA

(26) Assuming a density of one, the centroid of region R is $\left(\frac{\int_0^2 x(f(x)-g(x))dx}{\int_0^2 (f(x)-g(x))dx}, \frac{\int_0^2 \left(\frac{f(x)+g(x)}{2}\right)(f(x)-g(x))dx}{\int_0^2 (f(x)-g(x))dx}\right)$. Find this point.

(a) $\left(\frac{7}{5}, 1\right)$ (b) $\left(1, \frac{7}{5}\right)$

(c) $\left(\frac{5}{7}, 1\right)$ (d) $\left(1, \frac{5}{7}\right)$ (e) NOTA

(27) If, for continuous $f(x)$, $\int_a^d f(x)dx$ =10, $\int_a^c f(x)dx$ =7, and $\int_b^d f(x)dx$ =8, find $\int_b^c f(x)dx$

(a) 5 (b) −5

(c) 7 (d) −7 (e) NOTA

(28) Find the slope of the tangent line to $f(x) = \int_x^{x^2} e^{t^3} dt$ at $x = \sqrt{e}$.

(a) $e^e\left(2e^{e^2+0.5} - 1\right)$ (b) $2e^{e^3-e+0.5}$

(c) $2e^{e^3+0.5}$ (d) $2e^{e^3+0.5} - e^e$ (e) NOTA

(29) Approximate the area between $f(x) = \sin(x)$ and the x-axis from $x = 0$ to $x = \pi$ via Left-Handed Riemann Sum using six equal-width rectangles.

(a) $\frac{(1+\sqrt{3})}{6}\pi$ (b) $(1 + \sqrt{3})$

(c) $\frac{(2+\sqrt{3})}{6}\pi$ (d) $(2 + \sqrt{3})$ (e) NOTA

(30) Francisco and Ryan are the last two competitors in an Integration Bee, during which they must answer increasingly difficult integrals. Because they are both Calculus Masters™, there are infinitely many integrals on the test. For the n^{th} integral, the probability of Francisco getting it wrong is $\int_0^1 x^{n^2-1}dx$, and the probability of Ryan getting it wrong is $\int_1^2 \log_{n+1}(\sqrt[n]{x})\,dx$. Who is expected to get fewer integrals wrong, and therefore win?

(a) $Ryan$ (b) $Francisco$

(c) $They\ Tie$ (d) $Cannot\ Be\ Determined$ (e) NOTA