Nationals 2015 Mu Integration – ANSWERS

(1)	C
(2)	A
(3)	B
(4)	C
(5)	E
(6)	D
(7)	A
(8)	C
(9)	A
(10)	B
(11)	D
(12)	A
(13)	C
(14)	C
(15)	D
(16)	C
(17)	B
(18)	B
(19)	A
(20)	E
(21)	B
(22)	C
(23)	A
(24)	D
(25)	C
(26)	B
(27)	A
(28)	E
(29)	C
(30)	B

Nationals 2015 Mu Integration - SOLUTIONS

(1) **Solution:** $\int_0^1 e^2 dx = [e^2 x]_0^1 = e^2$. C

(2) **Solution:** $\int_0^1 (2x^3 - x + 1)dx = \left[\frac{1}{2}x^4 - \frac{1}{2}x^2 + x\right]_0^1 = 1$. A

(3) **Solution:** $\int_0^{\frac{\pi}{2}} \cos(3x)\, dx = \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{2}} = -\frac{1}{3}$. B

(4) **Solution:** $\int_{-1}^{1} |x| dx = \int_0^1 x dx + \int_{-1}^{0} -x dx = \left[\frac{1}{2}x^2\right]_0^1 + \left[-\frac{1}{2}x^2\right]_{-1}^{0} = \left(\frac{1}{2} - 0\right) + \left(0 - -\frac{1}{2}\right) = 1$. C

(5) **Solution:** $\int_1^5 \frac{1}{x-2} dx = \int_1^2 \frac{1}{x-2} dx + \int_2^5 \frac{1}{x-2} dx = \lim_{c\to 2}[\ln(|x-2|)]_1^c + \lim_{c\to 2}[\ln(|x-2|)]_c^5$. Both of these limits diverge, hence so does the integral. E

(6) **Solution:** Let $u = x^2 \to du = 2xdx \to \int_0^{\sqrt{\ln(2)}} xe^{x^2} dx = \int_0^{\ln(2)} \frac{1}{2} e^u du = \left[\frac{1}{2}e^u\right]_0^{\ln(2)} = 1 - \frac{1}{2} = \frac{1}{2}$. D

(7) **Solution:** Let $u = x^2 - 2x - 3 \to du = 2(x-1)dx \to \int_0^1 \frac{x-1}{x^2-2x-3} dx = \int_{-3}^{-4} \frac{\frac{1}{2}}{u} du = \left[\frac{1}{2}\ln|u|\right]_{-3}^{-4} = \frac{1}{2}\ln(4) - \frac{1}{2}\ln 3 = \ln\left(\frac{2}{\sqrt{3}}\right) = \ln\left(\frac{2\sqrt{3}}{3}\right)$. A

(8) **Solution:** $\int_0^1 \frac{1}{x^2-2x-3} dx = \int_0^1 \frac{1}{(x-3)(x+1)} dx = \int_0^1 \frac{\frac{1}{4}}{(x-3)} - \frac{\frac{1}{4}}{(x+1)} dx = \frac{1}{4}[\ln|x-3| - \ln|x+1|]_0^1 = \frac{\ln(2)-\ln(2)-\ln(3)+\ln(1)}{4} = -\frac{\ln(3)}{4}$. C

(9) **Solution:**
$\int_{-1}^{1} \frac{1}{x^2+2x+5} dx = \int_{-1}^{1} \frac{1}{(x+1)^2+4} dx = \left[\frac{1}{2}\arctan\left(\frac{x+1}{2}\right)\right]_{-1}^{1} = \frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0) = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$. A

(10) **Solution:** Let $u = \sqrt{1-\sqrt{x}} \to x = (1-u^2)^2 \to dx = -4u(1-u^2)du \to \int_0^1 \sqrt{1-\sqrt{x}} dx = -4\int_1^0 u^2(1-u^2)du = 4\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 = 4 \cdot \frac{2}{15} = \frac{8}{15}$. B

(11) **Solution:** The integral of an odd function on symmetric bounds is zero. D

(12) **Solution:** $\int_0^{\frac{\pi}{8}} \sec(x)\, dx = [\ln|\sec(x) + \tan(x)|]_0^{\frac{\pi}{8}} = \ln\left(\frac{4}{\sqrt{2}+\sqrt{6}} + \frac{-\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}}\right)$

$= \ln\left(4 - \sqrt{2} + \sqrt{6}\right) - \ln\left(\sqrt{2} + \sqrt{6}\right)$. A

(13) **Solution:** $y=\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}\to y=\sqrt{x+y}\to y^2-y-x=0\to y=\frac{1\pm\sqrt{1+4x}}{2}$. The integrand is positive, so $y=\frac{1+\sqrt{1+4x}}{2}\to\int_0^2\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}\,dx=\int_0^2\frac{1+\sqrt{1+4x}}{2}\,dx=\left[\frac{1}{2}x+\frac{1}{12}(1+4x)^{\frac{3}{2}}\right]_0^2=\left(1+\frac{9}{4}\right)-\frac{1}{12}=\frac{13}{4}-\frac{1}{12}=\frac{38}{12}=\frac{19}{6}$. C

(14) **Solution:** Using Integration by Parts twice, we get: $\int_0^1(x^2)(e^{2x}dx)=\left[(x^2)\left(\frac{1}{2}e^{2x}\right)\right]_0^1-\int_0^1\left(\frac{1}{2}e^{2x}\right)(2xdx)=\frac{1}{2}e^2-\int_0^1(x)(e^{2x}dx)=\frac{1}{2}e^2-\left(\left[(x)\left(\frac{1}{2}e^{2x}\right)\right]_0^1-\int_0^1\left(\frac{1}{2}e^{2x}\right)(dx)\right)=\frac{1}{2}\int_0^1e^{2x}dx=\frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_0^1=\frac{1}{4}e^2-\frac{1}{4}$. C

(15) **Solution:** Using Integration by Parts twice, we get:

$I\equiv\int_0^\infty(e^{-\alpha x})(\sin(x)\,dx)=[(e^{-\alpha x})(-\cos(x))]_0^\infty-\int_0^\infty(-\cos(x))(-\alpha e^{-\alpha x}dx)=$
$1-\alpha\int_0^\infty(e^{-\alpha x})(\cos(x)\,dx)=1-\alpha\left([(e^{-\alpha x})(\sin(x))]_0^\infty-\int_0^\infty(\sin(x))(-\alpha e^{-\alpha x}dx)\right)=1-$
$\alpha^2\int_0^\infty(e^{-\alpha x})(\sin(x)\,dx)=1-\alpha^2I\to I=\frac{1}{\alpha^2+1}$. D

(16) **Solution:** This is just the area of a quarter circle of radius 2. π. C

(17) **Solution:** Using the trigonometric substitution $x=\sec(\theta)\to\sqrt{x^2-1}=\tan(\theta)$ & $dx=\sec(\theta)\tan(\theta)d\theta$ gives
$\int_1^{\sqrt{2}}\frac{\sqrt{x^2-1}}{x}dx=\int_0^{\frac{\pi}{4}}\frac{\tan(\theta)}{\sec(\theta)}\sec(\theta)\tan(\theta)d\theta=\int_0^{\frac{\pi}{4}}\tan^2(\theta)\,d\theta=\int_0^{\frac{\pi}{4}}(\sec^2(\theta)-1)\,d\theta=[\tan(\theta)-\theta]_0^{\frac{\pi}{4}}=\left(1-\frac{\pi}{4}\right)-(0-0)$. B

(18) **Solution:** $\int_1^{\ln(2)}\ln\left(x^{e^x}\cdot e^{\frac{e^x}{x}}\right)dx=\int_1^{\ln(2)}e^x\ln(x)+e^x\frac{1}{x}\,dx=\int_1^{\ln(2)}\frac{d}{dx}(e^x\ln(x))\,dx=[e^x\ln(x)]_1^{\ln(2)}=2\ln(ln(2))$. B

(19) **Solution:** Using Weierstrass substitution, $t=\tan\left(\frac{x}{2}\right)$, $\cos(x)=\frac{1-t^2}{1+t^2}$ and $dx=\frac{2}{1+t^2}dt$. So $\int_0^\pi\frac{1}{3+\cos(x)}dx=\int_0^\infty\frac{1}{3+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt=\int_0^\infty\frac{2}{3+3t^2+1-t^2}dt=\int_0^\infty\frac{1}{2+t^2}dt=\frac{1}{\sqrt{2}}\left[\arctan\left(\frac{t}{\sqrt{2}}\right)\right]_0^\infty=\frac{1}{\sqrt{2}}\frac{\pi}{2}=\frac{\sqrt{2}}{4}\pi$.
A

(20) **Solution:** The discontinuity as x=1/2 actually causes the integral to diverge. E

(21) **Solution:** $1=\int_1^a\frac{6}{x^4}dx=-\frac{2}{x^3}\Big|_1^a=-\frac{2}{a^3}+2\to a=\sqrt[3]{2}$. B

(22) **Solution:** $\lim_{n\to\infty}\sum_{k=1}^n\frac{n}{(n+k)^2}=\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n}\frac{1}{\left(1+\frac{k}{n}\right)^2}=\int_1^2\frac{1}{x^2}dx=\left[-\frac{1}{x}\right]_1^2=\frac{1}{2}$. C

(23) **Solution:** $x = -x(x-3) \to x^2 - 2x = 0 \to x = 0 \,\&\, 2.$ $\int_0^2 (-x^2 + 3x - x)\,dx = \left[-\frac{1}{3}x^3 + x^2\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$ A

(24) **Solution:** $V = \pi \int_0^2 (-x^2 + 3x)^2 dx - \pi \int_0^2 (x)^2 dx = \pi \int_0^2 (x^4 - 6x^3 + 9x^2) dx - \pi \int_0^2 (x)^2 dx = \pi \left[\frac{1}{5}x^5 - \frac{3}{2}x^4 + 3x^3\right]_0^2 - \pi \left[\frac{1}{3}x^3\right]_0^2 = \pi \left(\frac{32}{5} - 24 + 24 - \frac{8}{3}\right) = \frac{96-40}{15}\pi = \frac{56}{15}\pi.$ D

(25) **Solution:** $2\pi \int_0^2 x(-x^2 + 2x) dx = 2\pi \left[-\frac{1}{4}x^4 + \frac{2}{3}x^3\right]_0^2 = 2\pi \left(-4 + \frac{16}{3}\right) = \frac{8}{3}\pi.$ C

(26) **Solution:** These integrals differ only by a constant from those calculated in previous questions. From (25), without the 2π, is $\int_0^2 x\big(f(x) - g(x)\big) = \frac{4}{3}.$ Since $\int_0^2 \left(\frac{f(x)+g(x)}{2}\right)\big(f(x) - g(x)\big) dx = \frac{1}{2}\left(\int_0^2 \big(f(x)\big)^2 dx - \int_0^2 \big(g(x)\big)^2 dx\right)$, we replace π with $\frac{1}{2}$ in (24) to get $\frac{28}{15}$. Finally the denominators are the area from (23) $\int_0^2 \big(f(x) - g(x)\big)\,dx = \frac{4}{3}.$ So $\left(\frac{\int_0^2 x\big(f(x)-g(x)\big)dx}{\int_0^2 \big(f(x)-g(x)\big)dx}, \frac{\int_0^2 \left(\frac{f(x)+g(x)}{2}\right)\big(f(x)-g(x)\big)dx}{\int_0^2 \big(f(x)-g(x)\big)dx}\right) = \left(\frac{\frac{4}{3}}{\frac{4}{3}}, \frac{\frac{28}{15}}{\frac{4}{3}}\right) = \left(1, \frac{7}{5}\right).$ B

(27) **Solution:** $10 = \int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx = \left(\int_a^c f(x)dx - \int_b^c f(x)dx\right) + \int_b^c f(x)dx + \left(\int_b^d f(x)dx - \int_b^c f(x)dx\right) = 7 + 8 - \int_b^c f(x)dx \to \int_b^c f(x)dx = 15 - 10 = 5.$ A

(28) **Solution:** $\frac{d}{dx}\left[\int_x^{x^2} e^{t^3} dt\right]\Big|_{x=\sqrt{e}} = 2xe^{x^6} - e^{x^3}\Big|_{x=\sqrt{e}} = 2\sqrt{e}e^{e^3} - e^{e^{1.5}} = 2e^{e^3+0.5} - e^{e^{1.5}}.$ E

(29) **Solution:** $A \approx \frac{\pi}{6}\left(\sin(0) + \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{5\pi}{6}\right)\right) = \frac{\pi}{6}\left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{(2+\sqrt{3})}{6}\pi.$ C

(30) **Solution:** $\int_0^1 x^{n^2-1}dx = \left[\frac{1}{n^2}x^{n^2}\right]_0^1 = \frac{1}{n^2}$ so Francisco's expected number correct is $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$ $\int_1^2 \log_{n+1}(\sqrt[n]{x})\,dx = \left[\frac{x\ln(x)-x}{\mathrm{n}\cdot\ln(n+1)}\right]_1^2 = \frac{\ln 4 - 1}{\mathrm{n}\cdot\ln(n+1)}.$ The sum $\sum_{n=1}^{\infty} \frac{\ln 4-1}{\mathrm{n}\cdot\ln(n+1)}$ diverges, so Ryan is expected to get infinitely many wrong. Therefore, Francisco is expected to win. B