Multivariable Calculus – Solutions

1. A - ∇f does not equal (0,0); f_x = 1at the point of interest.

2. D - The second derivative test is inconclusive for this function. By further inspection we see that f(x, 0) = x^3, which can become positive or negative as one moves away from (0,0). (0,0), therefore, cannot be a local max or min, but rather a saddle point.

3. C – Note that the function can be written as $(x+y)^2$. We know without calculus that this function cannot take negative values and so is minimized at (0,0).

4. B – Change the order of integration to enable evaluation. Graphing the region in question will help to decide the new bounds. The integral becomes: $\int_0^{\pi/2}\int_0^{y}\frac{\sin y}{y}\,dxdy$. Integrate the inside first, yielding $\int_0^{\pi/2}\sin y\,dy$, which equals 1.

5. C – $\overrightarrow{AB}$ = (1, 0, -1), $\overrightarrow{AC}$ = (5, 3, -2). Take the cross product of the two vectors to find the vector normal to the plane. $\overrightarrow{AB}\times\overrightarrow{AC}$ = (3, -3, 3), which points in the same direction as (1, -1, 1). The equation of the plane, therefore, becomes x – y + z = Z. Plugging in A, B, or C reveals that the constant Z = 3, and so the final equation is x-y+z=3. This is already in the required form, with a = -1, b =1, and c = 3.

6. A – This question is very quick if #5 is solved correctly. A line orthogonal to the plane will be in the direction of the plane's normal, or (1, -1, 1). A line in this direction passing through D can be represented by s(t) = (1, -1, 1)t + (-1, 4, -1). In a slightly different form, s(t) = (t-1, 4-t, t-1). There are other possible representations of this line, but of the given choices, only A is correct.

7. A – This equality is incorrect. The correct form, which would have been a form of Green's Theorem, is $\int_C Pdx + Qdy = \int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$

8. B – The first derivative gives velocity: (t^2-1, 2t, 0). The second derivative gives the desired acceleration: (2t, 2, 0). When t = 1, this becomes (2, 2, 0).

9. E(5) – We already calculated velocity as (t^2-1, 2t, 0). Speed is the magnitude of this vector, so $\sqrt{t^4+2t^2+1}$ = t^2+1. When t =2, speed is 5.

10. D – Arc length is calculated as the integral of speed over a time interval. In our case, it is $\int_{-2}^{2} t^2 + 1\,dt$= 28/3.

11. C – $0 \le \left|\frac{x^3y^3}{x^4+y^2}\right| \le \frac{|x^3y^3|}{y^2} = |x^3y|$ → 0, so the limit is 0.

12. A – If you fix y=0, the limit is 0. Fix y = x^2, the limit is ½. Since it approaches multiple values, the limit does not exist.

13. B – An easily provable theorem states this property of line integrals over orientation-reversing paths.

14. E(0) – Clairaut's Theorem guarantees the equivalence of mixed partials as long as the partials are continuous.

15. B – This can be visualized as a cylinder when unwound, with circle radius 1 and height 6π (the circumference of the circle of rotation). The volume is thus $6\pi^2$. Pappus' Theorem is the more technical form of this logic.

16. B – This is Fubini's Theorem

17. D – In the two-dimensional case, curl = $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. Here, $\frac{\partial Q}{\partial x} = e^{x+y} = e$ at (0,1). $\frac{\partial P}{\partial y} = xe^{xy} = 0$. Thus, curl is $e - 0 = e$. Because the curl does not equal zero at all points, the vector field cannot be a gradient vector field.

18. C – ∇ F(x,y,z) = (siny, xcosy + sinz, ycosz) = (0, -1, π) at the stated point. The plane becomes: – (y- π) + πz = 0, which simplifies to $y - \pi z = \pi$.

19. C – This is an application of the Implicit Function Theorem.

20. B – $\frac{\partial F}{\partial x} = \sin y + y\cos z\frac{\partial z}{\partial x} = 0$. Therefore, $\frac{\partial z}{\partial x}(1, \pi) = 0/\pi = 0$.

21. B – First calculate A^n; after multiplying out a few terms, the pattern becomes clear. $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. As n approaches infinity, the 0 and 1 entries will approach 0, but the n entry will approach 1. The resulting matrix is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the sum of the entries is 1.

22. A – $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$. Expanding along the bottom row and simplifying yields the concise answer of $-\rho^2\sin\varphi$, and the absolute value is just the positive. However, we could have known this without any calculation, as $\rho^2\sin\varphi$ is the "change of variables" factor used when integrating in spherical coordinates.

23. C – $\vec{a} \times (\vec{a} \times \vec{b}) = \vec{a}(\vec{a} \cdot \vec{b}) - \vec{b}(\vec{a} \cdot \vec{a}) = -2\vec{a} - \|\vec{a}\|^2\vec{b} = 0 \rightarrow \vec{a} = (\|\vec{a}\|^2/2)\vec{b}$. Since $\vec{a}$ is some scalar multiple of $\vec{b}$, the two vectors must be parallel and so $\vec{a} \times \vec{b} = 0$.

24. A – Beginning with the equality $\vec{a} = (\|\vec{a}\|^2/2)\vec{b}$ found in #23, we know that $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2 = \frac{\|\vec{a}\|^4}{4}(\vec{b} \cdot \vec{b}) = \frac{\|\vec{a}\|^4}{4}$. Taking the square root of both sides gives $\|\vec{a}\| = \|\vec{a}\|^2/2$, so $\|\vec{a}\| = 2$ and so $\vec{a} = -2\vec{b}$.

$\vec{a} \cdot \vec{c} = -2\vec{b} \cdot \vec{c} = -4$.

25. D – There are two constraints: $g_1(x,y,z) = x^2 + y^2 - 2 = 0$, and $g_2(x,y,z) = x + z - 1 = 0$. We must find x, y, z, λ_1, λ_2 such that $\nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z)$ and $g_1(x,y,z) = 0$, $g_2(x,y,z) = 0$.
After computing the gradients, we get the 5 equations:
$1 = \lambda_1(2x) + \lambda_2$; $1 = \lambda_1(2y)$; $1 = \lambda_2$; $x^2+y^2=2$; $x + z = 1$.
We solve to get x = 0, y = $\pm\sqrt{2}$, z = 1.

26. E ($f = x^2yz - \cos x + C$) – We know this is possible because the curl of the vector field is zero. Integrating the first term with respect to x yields $x^2yz - \cos x$ + l(y,z) + m(y) + n(z) + C. But, the l, m, and n terms are actually 0, as seen when taking the derivative with respect to y and z.

27. E – This question is a matter of basic integration done 3 times. First evaluating the inner integral, we get $\int_0^1 \int_0^x y^2 + \frac{xy^2}{2} dydx$. Again working from the inside, we simplify to $\int_0^1 \frac{x^3}{3} + \frac{x^4}{6} dx$ = (1/12) + (1/30) = 21/180.

28. B – Let $u = x - y$ and $v = y\sqrt{2}$. Then we have the equation $u^2 + v^2 \leq 1$, a circle with area π. To find the area in terms of the original variables, however, we must multiply this by the Jacobian: $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ $= \frac{1}{\sqrt{2}}$, giving the answer of $\frac{\pi\sqrt{2}}{2}$.

29. D – C bounds the surface S defined by z = 1 – x – y = f(x,y) for (x,y) in D = $\{(x,y)|x^2+y^2 \leq 1\}$.
Set **F** = $-y^3$**i** + x^3**j** – z^3**k**, which has curl **∇** x **F** = $(3x^2 + 3y^2)$**k**. By Stokes' Theorem, the desired line integral equals the surface integral $\int_S$ **∇** x **F** $\cdot\, d\mathbf{S}$ = $\int_D (3x^2 + 3y^2)dxdy$.
We can change this to polar coordinates, yielding: $3\int_0^1 \int_0^{2\pi} r^2 \cdot rd\theta dr$ = $6\pi \int_0^1 r^3 dr$ = $3\pi/2$.

30. C – We need a vector field such that **F·n** = $x^2 + y + z$. At any point on the ball, the outward unit normal **n** to ∂W is **n** = x**i** + y**j** + z**k**, since on ∂W, $x^2 + y^2 + z^2 = 1$ and the radius vector is normal to the sphere.
We therefore see that our vector field **F** = x**i** + **j** + **k**. div **F** = 1+0+0 = 1.
By the divergence theorem, $\int_{\partial W}(x^2 + y + z)dS$ = $\int_W dV$ = volume(W) = $\frac{4\pi}{3}$.