1. C
2. D
3. C
4. D
5. C
6. D
7. B
8. B
9. C
10. D
11. B
12. D
13. C
14. A
15. A
16. D
17. B
18. C
19. B
20. B
21. D
22. E
23. D
24. C
25. B
26. B
27. C
28. A
29. D
30. A
1. C The polynomial factors to \((2x - 1)(x + 2)(x - 2)\), so \(x = \frac{1}{2}, -2, 2\), but 2 is the only positive solution.

2. D Over \(\mathbb{Z}_7\), the polynomial can be factored to \((2x - 1)(x^2 - 4)\). So \(2x \equiv 1 \text{ or } x^2 \equiv 4 \mod 7\). The solutions are \(x = 4, 2, 5\).

3. C The question is equivalently asking \(19x \equiv 13 \mod 29\), which can be rewritten as \(19x = 29a + 13\). Taking both sides modulo 19 to get \(0 \equiv 10a + 13 \mod 19\), or \(19b = 10a + 13\). Taking both sides modulo 10 to get \(9b \equiv 3 \mod 10\). At this point, \(b = 7\), substituting back to get \(a = 12\) and \(x = 19\).

4. D Subtracting 5 from both sides of the original equation, \(6x \equiv 18 \mod 10\), but since \(\gcd(6, 10) = 2\), that yields \(x \equiv 3 \mod 5\). \(23 + 28 + 33 + 38 + 43 + 48 = 213\).

5. C Given that \(3x + 5y \equiv 0 \mod 37\), we have \(3x \equiv -5y \mod 37\), so \(40x \equiv -5y \mod 37\), or \(y \equiv -8x \mod 37\). Then \(ax + 7y \equiv ax - 56y \mod 37\). For that to always be divisible by 37, we have \(a \equiv 19 \mod 37\).

6. D \(x\) must divide into the pair-wise difference of the numbers given. \(87937 - 59117 = 28820\), and \(131167 - 87937 = 43230\). Therefore, the largest value of \(x\) is \(\gcd(28820, 43230) = 14410\). The sum of the digits is 10.

7. B Getting common denominator on the left, \(\frac{a+b}{ab} = \frac{1}{12}\). Cross multiply and rearrange to get \(ab - 12a - 12b = 0\), adding 144 to both sides and factor to get \((a - 12)(b - 12) = 144\). \(144 = 2^4 \cdot 3^2\), so it has 15 factors. \(a - 12\) can be equal to each of the 15 factors, with a corresponding value for \(b - 12\).

8. B Taking the entire equation modulo 13 to get \(5y \equiv 4 \mod 13\). By inspection, \(y \equiv 6\). This yields two solutions, \((55, 6)\) and \((24, 19)\). \(55 + 6 + 24 + 19 = 104\).

9. C Based on the information given, \(x = 16A + 4B + C\) and \(y = 36A + 6B + C\), so \(32A + 8B + 2C = 36A + 6B + C\), or \(4A - 2B - C = 0\). Since \(4A\) and \(2B\) are both even, \(C\) must also be even. If \(C = 0\), then \(B = 2, A = 1\), and \(x = 24\). If \(C = 2\), there are two possibilities as \(B\) can be either 1 or 3. \(A\) is equal to 1 or 2, respectively. So the two values for \(x\) are 22 and 46. Therefore, \(z = 92\), and sum of the digits is 11.

10. D Note that the norm of a Gaussian integer is the square of the more familiar complex norm. The norm of a product is simply the product of the norms. So the norm is \(5 \cdot 8 \cdot 50 \cdot 10 = 20000\).
11. B A real number is a Gaussian prime if and only if it is a prime that is 3 modulo 4. Otherwise, it is a sum of squares, and can be expressed as a product of two conjugates. For example, \(2 = (1 + i)(1 - i), 5 = (2 + i)(2 - i).\) (Similarly for purely imaginary numbers.)

A Gaussian integer with both real and imaginary parts is prime if its norm is a prime number (call it \(p\)). Then if it is expressed as a product of two Gaussian integers, their norms must be 1 and \(p\). The norm of \(3 + 5i\) is 34, so it can be expressed as the product of two Gaussian integers with norms 2 and 17. One such expression is \((1 + i)(4 + i)\). 2 + \(3i, 2 + 5i,\) and 3 are the Gaussian primes in the set.

12. D All but the sets of real numbers and the set of irrational numbers are countably infinite.

13. C \(720 = 2^4 \cdot 3^2 \cdot 5\), so \(a\) and \(b\) must only have 2, 3, and 5 in its prime factorization. Consider powers of 2 for \(a\) and \(b\). One of them must have \(2^4\) in its prime factorization, or their \(\text{lcm}\) cannot have \(2^4\). The other can have any power of 2 from 0 to 4. So the number of ways for \(a\) and \(b\) to have powers of 2 is \(2 \cdot 5 - 1 = 9\). The case where both have \(2^4\) is double counted, so it must be subtracted out. Similarly, there are \(2 \cdot 3 - 1 = 5\) ways for powers of 3, and \(2 \cdot 2 - 1 = 3\) ways for powers of 5. All 3 are independent of each other, making a total of \(9 \cdot 5 \cdot 3\) ordered pairs.

14. A Based on the information given,

\[
\frac{b^2 + 3}{b^2 + 3b + 6} = \frac{b + 4}{b + 8}
\]

Cross multiply to get \(b^3 + 8b^2 + 3b + 24 = b^3 + 7b^2 + 18b + 24,\) or \(b^2 - 15b = 0\).

Therefore, \(149_{15} = 225 + 60 + 9 = 294\) and \(338_{15} = 675 + 45 + 8 = 728. \frac{294}{728} = \frac{21}{52} = \left(\frac{16}{37}\right)_{15}.

15. A It’s fastest to consider this problem in chunks of 30. For every 30 consecutive integers, there are 15 multiples of 2, 10 multiples of 3, 5 multiples of 6, 3 multiples of 10, 2 multiples of 15, and 1 multiple of 30. So the number of integers satisfying the conditions given is \(15 + 10 - 5 - (3 + 2 - 1) = 16\). From 7 to 2016, there are 67 \cdot 16 = 1072\) such numbers, then tack on 2, 3, 4, and 6 for 1076 such numbers.

16. D Euler’s totient theorem does not apply directly, since \(\gcd(8, 800) \neq 1\). So we must consider \(8^{321}\) modulo 32 and 25, since \(800 = 2^5 \cdot 5^2\). Clearly, \(8^{321} \equiv 0 \mod 32\).

\(\phi(25) = 20\), so \(8^{20} \equiv 1 \mod 25\), or \(8^{321} \equiv (8^{20})^{16} \cdot 8 \equiv 8 \mod 25\). Now we are seeking a number less than 800 that is 0 mod 32 and 8 mod 25, and that number is 608.

17. B Let \(x = p^aq^b\), where \(p, q\) are distinct primes. Then \(N = (a + 1)(b + 1)\).

\(x^2 = p^{2a}q^{2b}\), so \(3N = (2a + 1)(2b + 1)\).

Therefore, \(3ab + 3a + 3b + 3 = 4ab + 2a + 2b + 1\),

Or \(ab - a - b = 2\). Adding 1 to both sides, \((a - 1)(b - 1) = 3\).

Since both \(a\) and \(b\) are positive integers (or \(x\) wouldn’t have two prime factors), one of them is 2, the other is 4, and \(N = 15\).

\(x^7 = p^{14}q^{28}\), which has \((14 + 1)(28 + 1) = 29N\) factors.
18. C For \( x^k \equiv k \mod 363 \) to be true in general, \( x^2 \equiv x \mod 363 \). So \( x^2 - x = x(x - 1) \) must be divisible by 363. Since 363 = \( 3 \cdot 11^2 \), either one of \( x \) and \( x - 1 \) is divisible by 363, or one of them is divisible by 121 and the other is divisible by 3. This leaves 4 possibilities modulo 363, which are \( x \equiv 0, 1, 121, 243 \mod 363 \). There are 3 of each of the 4 possibilities, since 243 + 363 \( \cdot 2 < 1000 \). Their sum is \( 3(0 + 1 + 121 + 243) + 4(363 + 2 \cdot 363) = 5451 \). \( 5^2 + 4^2 + 5^2 + 1^2 = 67 \).

19. B By Wilson’s theorem, \( (p - 1)! \equiv -1 \mod p \). So \( r(f(n), n) = n - 1 \) if \( n \) is prime. If \( n \) is not prime, \( r(f(n), n) = 0 \), since \( n \) is part of \( (n + 1)! \). So \( \sum_{n=1}^{25} r(f(n), n) = 91 \). You may remember that sum of primes under 25 is 100, and there are 9 of them. Alternatively, just add them up. 91 leaves a remainder of 1 when divided by 5.

20. B \[
\frac{9}{34} = \frac{1}{3 + \frac{7}{9}} = \frac{1}{3 + \frac{1}{\frac{9}{7}}} = \frac{1}{3 + \frac{1}{1 + \frac{2}{7}}} = \frac{1}{3 + \frac{1}{1 + \frac{1}{7}}} = \frac{1}{3 + \frac{1}{1 + \frac{1}{2}}}
\]
So \( \frac{9}{34} = [0; 3, 1, 3, 2] \). \( 3^2 + 1^2 + 3^2 + 2^2 = 23 \).

21. D Let \( x = [2; 1, 2] \). Then
\[
x = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]
Substituting \( x \) for the bolded part, we have
\[
x = 2 + \frac{1}{x + 1} = 2 + \frac{x}{x + 1} = \frac{3x + 2}{x + 1}
\]
So \( x^2 + x = 3x + 2 \), or \( x^2 - 2x = 2 \).
Completing the square to get \( (x - 1)^2 = 3 \), and since \( 2 < x < 3 \), \( x = 1 + \sqrt{3} \).

22. E Convergents are the truncated continued fractions. Note that \( 3 = 3, \frac{22}{7} = 3 + \frac{1}{7} \), and \( \frac{355}{113} = 3 + \frac{1}{\frac{1}{7} + \frac{1}{15 + \frac{1}{1}}} \). So the third convergent is \( 3 + \frac{1}{\frac{1}{7} + \frac{1}{15 + \frac{1}{1}}} = \frac{333}{106} \).

23. D When the convergents are written as \( \frac{x}{y} \), the solutions to \( x^2 - 2y^2 = \pm 1 \), alternating between 1 and -1, starting with \( 1^2 - 2 \cdot 1^2 = -1 \). The first six convergents of \( \sqrt{2} \) are \( \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70} \). (99, 70) is the third smallest solution to \( x^2 - 2y^2 = 1 \).

24. C Each element \( r \) of the list is in the form of \( \frac{p}{q} \), where \( p \) is a factor of 5400 = \( 2^3 \cdot 3^3 \cdot 5^2 \), and \( q \) is a factor of 2880 = \( 2^6 \cdot 3^2 \cdot 5 \). So \( r = \pm 2^a \cdot 3^b \cdot 5^c \), where \( -6 \leq a \leq 3 \), \( -2 \leq b \leq 3 \), and \( -1 \leq c \leq 2 \), so the total number of possibilities is \( 2 \cdot 10 \cdot 6 \cdot 4 \), and with 6 roots, the probability of picking a root is 1/80.
25. B Dropping height from \( A \) to \( CD \), and call the foot \( F \), possibly coinciding with \( D \). Since \( m \angle \angle C < m \angle \angle D \), \( DF < CE \). Call, \( DF = x \), \( AD = a \), \( BC = b \), and height \( h \), not necessarily rational. Then \( CE = 6 - x \), and \( x \) can only be 0, 1, or 2. Applying Pythagorean Theorem to \( \triangle AFD \) and \( \triangle BCE \), \( x^2 + h^2 = a^2 \), \( (6 - x)^2 + h^2 = b^2 \). Subtracting the first from the second to arrive at \( 36 - 12x = (b - a)(b + a) \).

Now consider the cases:

1. \( x = 0 \), then \( (b - a)(b + a) = 36 \), for both \( a \) and \( b \) to be positive integers, \( b - a = 2 \) and \( b + a = 18 \), or \( b = 10, a = 8 \).
2. \( x = 1 \), then \( (b - a)(b + a) = 24 \), there are 2 subcases:
   a. \( b - a = 2, b + a = 12 \), then \( b = 7, a = 5 \)
   b. \( b - a = 4, b + a = 6 \), then \( b = 5, a = 1 \), but this makes the trapezoid degenerate, and thus is not a solution.
3. \( x = 2 \), then \( (b - a)(b + a) = 12 \), then \( b - a = 2, b + a = 6 \), or \( b = 4, a = 2 \).
   This is also degenerate.

This leaves a total of 2 possible trapezoids.

26. B Let \( x^3 = 31,217,193,218,303 \). Clearly, the unit digit of \( x \) must be 7. Eliminate E. Consider \( x^3 \) modulo 8. It is 7 mod 8, so \( x \) must be 7 mod 8. Eliminate A. \( x^3 \) is 8 mod 9, so \( x \) must be 2 mod 3. Eliminate C. Finally, \( 32087^3 > 32000^3 = 1024 \cdot 32 \cdot 10^9 > 32 \cdot 10^{12} \). Eliminate D.

27. C \( 30 = 2 \cdot 3 \cdot 5 \), and \( 54 = 2 \cdot 3^3 \). For \( 30|54N \), \( N \) must have a factor of 5. For \( 54|30N \), \( N \) must have factors of \( 3^2 \). Therefore, \( N \) must be a multiple of 45. Further, \( N \) divides 54 \cdot 30 = 1620, so \( N \) must also be a factor of 1620. 1620 = \( 2^2 \cdot 3^4 \cdot 5 \), reserving \( 3^2 \cdot 5 \) to ensure a multiple of 45 leaves \( 2^2 \cdot 3^2 \), so there are \((2 + 1)(2 + 1) = 9\) possibilities for \( N \).

28. A When substituting \( a \cdot b^n + c \cdot d^n \) into the recurrence, the powers of \( b \) and \( d \) stay separate, and \( a \) and \( c \) factor out, so simply substituting in \( b^n \) is sufficient for solving for \( b \) and \( d \). \( G_n = 3G_{n-1} + 4G_{n-2} \) becomes \( b^n = 3b^{n-1} + 4b^{n-2} \), or \( b^{n-2}(b^2 - 3b - 4) = 0 \). This yields two real solutions, which are the values of \( b \) and \( d \), and they add to 3.

Taking the recurrence back one step (using \( n = 2 \)), we have \( 13 = 3 \cdot 7 + 4G_0 \), or \( G_0 = -2 \). Substituting in 0 for \( n \) in the explicit formula to get \( a + c = -2 \).

So \( a + b + c + d = 1 \).

29. D \( 0.1_{10} = (0.00011)_2 \), so \( m = 1, n = 4 \), and \( \sum_{k=1}^{n} A_m k 2^k = 8 + 16 = 24 \).

30. A \( A = 4, B = 4, C = 2, D = 0 \), the sum is 10.