Mu Gemini Solutions

1. D: $x^4+4x^3+6x^2+4x+1=16=(x+1)^4$. Real solns are 1 and -3.

2. A: $4a^2=a/2, a=0,1/8$

3. B: $\frac{2050}{v}=\frac{260}{v-358}$

4. D: $405^{247}\equiv 1 \pmod 4, f^{(405^{207})}=f^{(1)}=\cos x$

5. C: $x^2+2x=\log_2 7, (x+1)^2=x^2+2x+1=\log_2 7+1$

6. D: $\sin(2000\pi x)\sin(2001\pi x)=\cos(2001\pi x-2000\pi x)-\cos(2001\pi x+2000\pi x)$,
$\int_0^1 \cos(4001\pi x)-\cos(\pi x)\,dx = sin(4001\pi x)-sin(\pi x)\mid_0^1=0$

7. A:

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{1/2\frac{5y}{2x+5y}}{1/2\frac{3x}{3x+2y}+1/2\frac{5y}{2x+5y}} \qquad \text{Simplify..}$$

8. A: The middle segment has a root at zero, so we have to integrate over $[-3,-1],[-1,0],[0,1],[1,3]$ which yields $8/3+1/4+1/4+8/3=35/6$.

9. A: $\cos(2x)=1-\sin^2 x, 1-2\cdot\frac{7}{36}=\frac{11}{18}$

10. B: A function is not invertible if there is a local minimum or maximum. For there to be no zeros in the first derivative, the discriminant $(2b)^2-4\cdot 48\cdot 3$ must be negative. This holds for $b<12$ (2,3,5,7,11).

11. C: The centroid is the average of the coordinates for the three vertices.

12. C: $f^{(n)}(0)$ is 0 for n even, 1 for n odd. $g^{(n)}(0)$ is 1 for n even, 0 for n odd. The Taylor series for $f(x)+g(x)$ around 0 is then the same series as for e^x, and $100e^1=271.828\ldots$

13. D: $56=\left[\binom{n-2}{5}+\binom{n-2}{4}\right]+\left[\binom{n-2}{4}+\binom{n-2}{3}\right]=\binom{n-1}{5}+\binom{n-1}{4}=\binom{n}{5}$

14. A: $\lim_{n\to\infty}(1-1/n)^n=1/e$

15. A: Solving the system yields prices of \$2.50, \$1.50, \$0.50 for a bag of chips, a soda, and a candy bar, respectively.

16. B: $a_1'=\cos\theta, a_{n+1}'=a_n'\cos a_n$. $a_2'=\cos a_1=0$, so by the Chain Rule, all a_n', $n\geq 2$ are 0.

17. B: $P(n,r)$ picks the r people, but we overcounted by the r different rotations around the table.

18. A: We want to minimize the total time $\frac{x+z}{y}+\frac{y}{x+z}=u+1/u$. The minimum occurs when $x+z=y$ with a total time of 2 hours.

19. E (650): $p=2, a=1, b=50$

20. C: If such a limit exists, then $\lim_{n\to\infty}a_{n+1}-a_n=0$. Solving $a_n=\frac{a_n^p(p-1)+c}{pa_n^{p-1}}$ for a_n yields $c^{1/p}$.

21. D: A midpoint will be a lattice point if the vector between the two endpoints consists of all even integers. There are 8 possibilities for parities of the components of a coordinate, so by the Pigeonhole Principle, if we have 9 components, there are two that share the same parity.

22. D: $\binom{5}{2}\cdot\binom{4}{2}\cdot\binom{3}{2}=180$

23. B: $\lim_{x\to a}(f(x)+g(x))^2-\lim_{x\to a}(f(x)-g(x))^2=\lim_{x\to a}4f(x)g(x)=3$.

24. A: The x^5th term of xP(x+3) is asking for the negative of the sum of the roots a polynomial containing the root 0 and the five roots of P shifted by -3. The second term is just triple the leading coefficient of one. $-(207-3(5))+3=-189$.

25. D: Reflect a pole so that it goes into the ground insted of toward the sky. The shortest path between the tips of the poles is a straight line, intersecting $\frac{4}{7}\cdot 240$ from the larger pole.

26. A: Consider the cross-section, inscribing a rectangle of maximal area in an isosceles triangle of base $2R$ and height H. A point on a leg of the triangle would look like $(x, H-H/R\cdot x)$. Solving for the maximum of $V=\frac{\pi}{3}x^2(H-H/R\cdot x)$ yields $x=2R/3$. Plugging back in gives the desired answer.

27. D: By inspection, $a_2=1$ and $a_3=1$, so for all $n\geq 1$, $a_n=1$.

28. D: $\frac{d}{dx}f^{-1}(x)=1/f'(f^{-}1(x))$. $f(x)=0$ at $x=2$, so $f^{-1'}(0)=1/(1/129)$.

29. E (735471): Give each child one candy bar. Then there are 16 left to be distributed to the kids in the traditional zero-or-more / stars-and-bars fashion: $\binom{16+9-1}{9-1}$

30. A: $P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{1/2\cdot 5/36}{1/2\cdot 5/36+1/2\cdot 21/216}=10/17$

31. D: The sine of an angle of a right triangle with adjacent leg x and opposite leg 1 is $1/\sqrt{x^2+1}$. The cosine of an angle of a right triangle with opposite leg 1 and adjacent leg $\sqrt{x^2+1}$ is $\sqrt{x^2+1}/\sqrt{x^2+2}$.

32. E (-2): $f'(x)=-2x\cdot \mathit{stuff}-2e^x\cos x$.

33. B: $\frac{1}{x^2+3x+2}=\frac{1}{(x+2)(x+1)}=\frac{1}{x+1}-\frac{1}{x+2}$, so the sum telescopes away, leaving $1/2-1/1002=250/501$.

34. D: $\int_0^{0.5}x/(1-x)\,dx=\int_0^{0.5}-1+1/(1-x)\,dx=-0.5+\ln 0.5$

35. A: Consider the cross-section along the diagonal of the cube. Let x be the length of an edge of the cube. The distance from the center of a face of the cube to a vertex of the cube is $x/\sqrt{2}$. The edge goes from the base of the cube to the cone, so the cross-section has similar triangles, producing the ratios $\frac{x/\sqrt{2}}{2}=\frac{5-x}{5}$. Solving for x yields $\frac{50\sqrt{2}-40}{17}$. Compute $6x^2$ to get the desired answer.

36. B: $\int_1^3\frac{x+1}{x+2}\,dx=\int_1^3 1-\frac{1}{x+2}\,dx=2-\ln 5+\ln 3$

37. C: In the limit, the ratio of consecutive Fibonacci numbers is the Golden Ratio, so the radius of convergence for the power series would be the reciprocal. A more robust argument is to use the closed-form representation of the Fibonacci series: $F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. The first term dominates, so we can ignore the second term. The series has to have a ratio of less than 1, so the radius of convergence is the reciprocal of the Golden Ratio.

38. B: Consider a revolution around the y-axis. The volume of revolution is $2\pi\int_a^b xf(x)\,dx$ and the x-coordinate of the centroid is $\frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}$. Operate on f^{-1} and the area remains the same and the axis of revolution switches.

39. C: Running Euclid's Algorithm on (C) yields: $(4x+5, 6x+7)\rightarrow(2x+2, 4+5)\rightarrow(1, 2x+2)$, indicating a greatest common divisor of 1.

40. B: $x=r\cos\theta, y=r\sin\theta, z=r\tan\alpha$ The determinant of the Jacobian for the change of variables is:

$$\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\alpha\\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\alpha\\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\alpha\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ \tan\alpha & 0 & r\sec^2\alpha\end{vmatrix}=r^2\sec^2\alpha$$

The integral is then $\iiint\frac{r^2\sec^2\alpha\,dr\,d\theta\,d\alpha}{r^2}$.