Mu Alpha Theta National Convention: Denver, 2001
Interschool Test

Each problem on this test is worth 12 points. If a problem consists of multiple parts, the score for the problem shall be $S = \left[12\left(\frac{c}{t}\right)\right]$, where *c* is the number of parts the team gets correct, *t* is the total number of parts, and [*x*] is the greatest integer less than or equal to *x*.

1. You are in the land of liars and truth-tellers, where each person you meet either always makes statements which are true, or always makes statements which are false. You come upon a group of four people, who make the following statements:
   A: C is a truth-teller
   B: D and C are opposite types of people
   C: B and A are opposite types of people
   D: At least one of A and B is a liar.
   List whether each of A, B, C, and D is a liar, a truth-teller, or cannot be determined.

2. Equiangular hexagons:
   A. An equiangular hexagon has sides of 1, *X*, 3, 4, 5, and 6, going clockwise. What is *X*?
   B. An equiangular hexagon has sides of 1, 2, 3, 4, 5, and 6. What order are these sides in?

3. Find the center of a sphere passing through the points (1, 1, 3), (2, 2, 1), (3, 4, 8), and (5, 6, 9).

4. Let $f(x,y) = \begin{cases} x - y & \text{if } x \text{ or } y < 0 \\ f(x-1,y) + f(x,y-1) & \text{otherwise} \end{cases}$
   What is the value of $f(10,9)$?

5. In this film, the character played by Mel Gibson showed a boy how to find the center of a circle given two of its chords.

6. The domain and range of a function $f(x)$ is the set of natural numbers. Moreover, f has the following properties for $y \geq 1$, what is the maximum value of f on the interval $1 \leq x \leq 2001$?
   $f(1) = 1$
   $f(2y) = f(y)$
   $f(2y+1) = f(2y) + 1$

7. According to psychologist George Miller, this is the average number of meaningful items that can be held in short-term memory at once.

8. Find the smallest positive integer *a* such that 2*a* + 1 and 6*a* + 4 are perfect squares.

9. Insert the missing term of the complete sequence: 822, _____, 49521214. Although many numbers fit the given sequence, this one should have the most significance to members of Mu Alpha Theta.

10. Eulerians, an alien race from a far-off galaxy, use an alphabet that consists only of the letters E, L, and R. How many eight-letter words in the Eulerian language contain an even number of Rs (including the case where there are none), assuming that all letter combinations produce a valid Eulerian word?

11. A father and his son are speaking. "My age now is XY, and yours is YX in base ten," says one. The other replies, "Tomorrow, my age is exactly twice yours!" Express as ordered pairs all possible values of the father's and son's ages. Ignore strange ways of counting birthdays that occur on leap days.

12. What is the perimeter of the smallest triangle with integer side lengths where one angle is three times the size of another?

13. Generate the number 24 from the following sets of four digits using only the mathematical symbols +, -, *, /, and parentheses. E.g. if the digits were 2223, you could do 2*2*2*3. Note that you may not combine two digits into a single number, so if the digits were 2411, you could not do 24*1*1. Also note that you may use the four digits in any order you wish.
    A. 9642
    B. 4133
    C. 7321
    D. 8477
    E. 9651
    F. 7641
    G. 8875
    H. 8566
    I. 8522
    J. 3444
    K. 9752
    L. 1555
    M. 8166
    N. 8833

14. Arrange the numbers 1-32 in a circle so that every pair of adjacent numbers sums to a square number.

15. Find a cycle of six 4-digit numbers such that the last 2 digits of each number are equal to the first 2 digits of the next number in the cycle. Each of the 6 numbers must be one of the following types (with all 6 types being represented): Square, Cube, Triangular, Prime, Fibonacci, Power-of-Two. Note: the same number may not appear more than once in the cycle.

16. What is the area of the smallest square of wrapping paper you need to wrap a unit cube without cutting the paper?

17. Professor Moody gives two of his students, Hermione and Ron, the product and sum (respectively) of two (not necessarily distinct) non-zero digits (1 to 9).

    Hermione says "I don't know the numbers".
    Ron says "I don't know the numbers".
    Hermione says "I don't know the numbers".
    Ron says "I don't know the numbers".
    Hermione says "I don't know the numbers".
    Ron says "I don't know the numbers".
    Hermione says "I know the numbers".

    What are the two non-zero digits Hermione now knows?

18. What is the smallest natural number that leaves remainders of 11, 2, and 44 when divided by 21, 37, and 51, respectively?

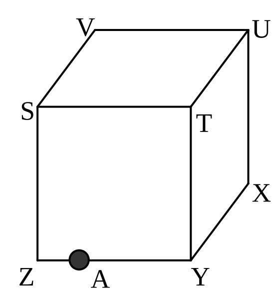
19. Julia is as old as John was when Julia's age was one quarter of six less than the age that John will be when Julia is John's current age. When John and Julia's present ages are added together, and the digits of the result are reversed, it is twice Julia's present age. Assuming John and Julia have the same birthday and thus the difference of their ages is always a constant, what is the absolute value of the difference between John's age and Julia's age?

20. A pirate ship lands near Bufords, in Bedford County, Virginia, and digs up Thomas J. Beale's treasure, which they convert into 100 Galleons (a form of pirate currency). The treasure has to be split among the 5 pirates: 1, 2, 3, 4, and 5 in order of rank, who are all infinitely smart, bloodthirsty, and greedy. Starting with pirate 5 (then 4, etc.) they can make a proposal how to split up the treasure. This proposal can either be accepted or the pirate is thrown overboard. A proposal is accepted if and only if a majority of the pirates agrees on it. If pirates 1, 2, 3, 4, and 5 receive A, B, C, D, and E coins respectively, what is the value of $A^2 + B^2 + C^2 + D^2 + E^2$?

21. Harry, Ron, Hermione, and Neville want to cross a bridge. They all begin on the same side. It is night, and they have only one flashlight with them. At most two people can cross the bridge at a time, and any party who crosses, either one or two people, must have the flashlight with them. The flashlight must be walked back and forth: it cannot be thrown, etc. Each person walks at a different speed. A pair must walk together at the speed of the slower person. Harry needs one minute to cross the bridge, Hermione needs two minutes, Ron needs five minutes, and Neville needs ten minutes. For example, if Harry and Ron walk across together, they need five minutes. What is the smallest number of minutes in which all four can get to the other side of the bridge?

22. Consider a grid of size 4 x 4 (i.e. sixteen squares), where all squares should get a color. The colored grid should meet the following conditions:
    4 squares should be colored blue,
    3 squares should be colored red,
    3 squares should be colored white,
    3 squares should be colored green,
    3 squares should be colored yellow, and
    no color appears more than once in any horizontal, vertical, or diagonal line.
    Fill in the grid on the answer sheet with the letters B, R, W, G, and Y to satisfy these conditions.

23. Some of the members of the math team decide to form the Calculator Club. In order to be exclusive, the requirements for being a member of the calculator club are that the number of calculators a member owns must be a perfect square. Also, each member must have a number of calculators which differs from another member's number of calculators by 192. If this club eventually grows to have its maximum possible number of members, determine the difference between the greatest number of calculators owned by a member and the least number of calculators owned by a member. The number of calculators owned by each member of the Calculator Club must be unique, i.e. two members may not own the same number of calculators.



24. An ant is on the surface of cube *STUVWXYZ* with edge length ten centimeters at point *A*, which is three centimeters from point *Z* along edge $\overline{ZY}$. Point *B* is another point on the cube's surface such that the minimum distance the ant must travel along the cube's surface to reach point *B* is a maximum. How far is point *B* from point *U*?

25. Given that *A*, *B*, and *C* are positive integers, and that
    GCF(*A*, *B*) = 300
    GCF(*B*, *C*) = 150
    GCF(*C*, *A*) = 6750
    LCM(*A*, *B*) = 135000
    LCM(*B*, *C*) = 81000
    LCM(*C*, *A*) = 202500
    evaluate *A* + *B* + *C*.

26. A person sits down and begins flipping a coin repeatedly. What is the probability that he will get a string of at least four heads at some point before he first gets a string of at least two tails?

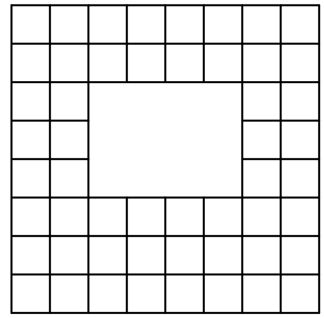
27. There exists point *A* on circle *O*. Points *E* and *F* are chosen randomly in the interior of circle *O*. What is the probability that *E* lies within triangle *OFA*?

28. Tetrahedron *ABCD* has edges $\overline{AB}, \overline{AC}, \overline{AD}, \overline{BC}, \overline{BD}$, and $\overline{CD}$ with lengths of $\sqrt{3}$, $\sqrt{6}$, 1, $\sqrt{5}$, $\sqrt{2}$, and $\sqrt{5}$ centimeters, respectively. What is the volume of *ABCD*, in cubic centimeters?

29. On Earth the year is divided into four seasons: spring, summer, autumn and winter. In the northern hemisphere it is summer when it is winter in the southern hemisphere, and the other way around. Which of the following sentences is true?
    A. The summer on the northern hemisphere is longer than the summer on the southern hemisphere.
    B. The summer on the northern hemisphere is shorter than the summer on the southern hemisphere.
    C. The summer on the northern hemisphere is as long as the summer on the southern hemisphere.

30. In a math contest, three problems (A, B, and C) were posed. Among the participants, there were 25 who solved at least one problem. Of all the participants who did not solve problem A, the number who solved problem B was twice the number who solved C. The number who solved only problem A was one more than the number who solved A and at least one other problem. Of all the participants who solved exactly one problem, half did not solve problem A. How many solved problem B and no other problems?

31. How many rectangles have sides determined by the gridlines in the figure?



32. Two points are simultaneously and randomly selected from a five by nine grid of 45 lattice points. What is the probability that the distance between them is a rational number?

33. How many paths of length nine are there on the surface of a 3 by 3 by 3 Rubic's Cube from one vertex to the vertex that is farthest from it? Each path must consist of edges of the component unit cubes.

34. Starting with the ordered quadruplet (0, 0, 0, 0), perform a series of "moves" in which you may add one to any one of the four numbers. Your goal is to produce the ordered quadruplet (3, 3, 3, 3). How many different ways can you do this?

35. How many positive six-digit numbers have the property that the units digit is the sum of the other five digits? Note that a number may not begin with a non-zero digit.

36. Imagine that you have a calculator and you are trying to produce the number 100 on the display. Use the digits 0, 1, 2, 3, 4, 5, and 6 and the operations +, $\times$, and / <u>*each exactly once*</u> (digits *and* operations) to produce the number 100. You may use as many parentheses as you like. For example, you could use the numbers 1, 2, 3, 4, 8 and the operations $\times$ and – each exactly once to produce $132 - 4 \times 8 = 100$.

37. How many positive integers less than 1000 have exactly six positive integer divisors?

38. How many circles in the plane pass through at least three of the points (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)?

39. A particle starts at the origin (0, 0) at time 0. It moves at one unit per second first to (1, 0), then to (1, 1), (0, 1), (-1, 1), and to (-1, 0), spiraling rectangularly outward in a counter-clockwise direction. Find the location of the particle after 2001 seconds.

40. Factor the expression: $12\left(a^2 + b^2 + c^2\right) + 26ab + 74bc + 51ca$

41. Dobby is an alien, of the Fleeshou race. Dobby needs to leave Home, a star-system located at (36, -6, 6) and go to Town, a star-system located at (16, -2, 18). However, Dobby's spaceship is low on hydrogen, which he must pick up "on the way" at The River, a nebula whose equation is $3x - 2y + z = 14$. What is the shortest distance Dobby can travel, satisfying these requirements?

42. What is the shortest distance between line *A*, which passes through the points (18, -2, -9) and (-6, -10, 7), and line *B*, which passes through the points (-31, -15, 12) and (-55, -18, 24)?

43. Determine all complex values of z for which $\cos(z) = 2$.

44. A game is played between two players which consists of individual rounds. Each round may be won by either of the players, and the total number of rounds won by each player is tracked. The game is finished when one of the players has won three more rounds than the other; the player who won the greater number of rounds is declared the winner. Players A and B play this game, with the probability of player B winning any given round being $\frac{3}{5}$. What is the probability that B wins this game?

45. How many distinct five-card poker hands can Steph be dealt from a non-standard 48-card deck (all four 3's have been removed), which can be straight flushes if 2's are regarded as wild cards (they may act as any other card desired)? A straight flush is five cards that are in the same suit, which are in sequential order ("4, 5, 6, 7, 8 of hearts" would be a straight flush, as would "10, J, Q, K, A of spades"). Note that Aces may only be the high card of a straight flush, not the low card. Because 2's are regarded as wild cards hands such as "6 of diamonds, 2 of clubs, 8 of diamonds, 2 of hearts, 10 of diamonds" will also count as straight flushes, because the wild 2's will "become" the 7 and 9 of diamonds.

46. Determine what is significant about the following complete sequence: 6, 2, 6, 7, 9, 7, 0, 7, 2, 7.

47. How many lattice points in the first quadrant satisfy the equation $7x + 11y = 500$?

48. How many non-congruent ellipses with axes of integer length have an area of 2000π?

49. Find the volume of the solid formed when the triangle with vertices (3, 5), (6, 17), and (15, 5) is revolved about the line $10x - 24y + 1 = 0$.

50. Let C be the hyperbola $25x^2 - 16y^2 = 400$ and P be the point $\left(5, \frac{15}{4}\right)$. Find the area of the parallelogram formed by C's asymptotes and the parallels to the asymptotes going through P.

51. Evaluate: $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$.

52. Mu Alpha Theta Trivia:
    A. What was the most recent National Convention to have an Open Division Topic Test named "Short Cuts"?
    B. In what year was the Sister Scholastica Award created?
    C. Where was the first national convention held?

53. Given that

$$\lim_{x\to z}(a(x)) = T\ ,$$
$$\lim_{x\to z}(b(x)) = S\ ,$$
$$\lim_{x\to z}(c(x)) = H\ ,$$
$$\lim_{x\to z}(d(x)) = K\ ,$$
$$\lim_{x\to z}(e(x)) = E\ \text{, and}$$
$$\lim_{x\to z}(f(x)) = Y\ ,$$

evaluate: $\lim_{x\to z}(a(x)b(x)c(x)d(x)e(x)f(x))$.

54. Determine a formula for the sum of the sixth powers of the first n natural numbers.

55. One hallway that is 4 feet wide meets another hallway that is 8 feet wide in a right angle. What is the length of the longest ladder that can be carried horizontally around the corner, to the nearest hundredth of a foot?

56. In ΔABC, BC = 8, AC = 15, and AB = 17. P is positioned inside ΔABC so that $\angle PAB = \angle PCA = \angle PBC$. Find $\tan \angle PCA$.

57. If F_n equals the nth Fibonacci number, evaluate $\sum_{n=1}^{\infty} \frac{F_n}{6^n} = \frac{1}{6} + \frac{1}{36} + \frac{2}{216} + \frac{3}{1296} + \ldots$

58. A piece of paper is in the shape of an equilateral triangle with a side length of 5 centimeters. Suppose the triangle is folded in such a way that one of its vertices lies along the side opposite to it, splitting that side into a 4:1 ratio. Find the length of the crease formed, in centimeters.

59. Find the product of all real solutions x to the equation $(2x-4)^4 + (9-3x)^4 = (x-5)^4$.

60. Four distinct points are chosen on the circumference of a circle. What's the probability they all lie within an arc with a measure of $\frac{\pi}{3}$ radians?

61.

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62. 9, 5, 8, 6, 17, 12, 19, 27, 29, 60, 56, 37, 44, 34, 14, 69, 46, 118, 26, 10, 126, 1006, 16, 18, 13, 11, 141, 38, 176, 123, 447, 563, 390, 214, 665, 116, 549, 266, 91, 390, 549, 521, 95, 825, 263, 582, 211, 440, 327, 380, 143, 312, 493, 136, 92, 127, 356, 85, 104, 446, 289, 242, 101, 115, 416, 185, 228, 575, 234, 99, 408, 32, 64, 42, 305, 135, 342, 216, 314, 599, 301, 1, 100, 101, 470, 27, 9, 3, 44, 16, 47, 21, 8, 56, 70, 1, 10, 126, 29, 6, 73, 10, 141, 176, 6, 9, 26, 54, 52, 20, 176, 185, 79, 143, 273, 493, 207, 119, 260, 74, 93, 1, 37, 12, 69, 9, 18, 17, 14, 4, 24, 26, 31, 238, 107, 35, 75, 105, 59, 164, 46, 30, 51, 242, 237, 152, 781, 334, 375, 582, 65, 85, 416, 198, 256, 159, 237, 184, 209, 317, 212, 141, 115, 32, 99 1, 5, 3, 17, 29, 6, 16, 12, 44, 4, 56, 52, 69, 70, 2, 55, 10, 8, 28, 47, 19, 32, 67, 45, 305, 402, 89, 100, 367, 57, 82, 416, 641, 101, 241, 375, 439, 74, 119, 244, 250, 45, 184, 237, 32, 67, 179, 51, 537, 402, 30, 35, 446, 20, 79, 93, 172, 115, 74, 493, 113, 65, 207, 439, 260, 416, 256, 470, 143, 825, 865, 92, 57, 99, 39, 148, 366, 575, 51, 537, 48, 110, 62, 59, 89, 25, 35, 108, 11, 44, 9, 496, 14, 132, 71, 31, 50, 126, 238, 87, 43, 153, 151, 20, 36, 35, 295, 112, 59, 46, 537, 89, 176, 178, 59, 300, 178, 112, 30, 42, 212, 23, 191, 81, 280, 825, 161, 237, 317, 39, 77, 301, 57, 256, 218, 78, 96, 74, 92, 439, 509, 493, 153, 244, 119, 249, 9, 11, 14, 19, 28, 132, 56, 31, 509, 52, 575, 35, 238, 13, 48, 25, 30, 176, 221, 164, 48, 125, 105, 209, 121, 51, 305, 237, 72, 317, 191, 77, 82, 78, 334, 159, 301, 85, 100, 340, 942, 416, 408, 104, 407, 88, 575, 143, 470, 493, 86, 463, 297, 78, 82, 77, 334, 159, 237, 513, 402, 90, 45, 305, 48, 300, 176, 55, 295, 156, 52, 11, 44, 24, 41, 70, 29, 18, 6, 5, 28, 83, 63, 59, 305, 575, 298, 86, 153, 96, 136, 45, 105, 75, 14, 9, 1, 44, 509, 141, 48, 305, 237, 364, 92, 127, 301, 616, 209, 300, 32, 51, 228, 77, 18, 119, 489, 549, 563, 956, 902

63. In this puzzle, you must fill in the grid below and to the right of the thick black lines. The grid is to be filled in with each square being either black or white. The numbers above a column indicate how many black squares appear in sequence in that column from top to bottom. If there is more than one number, that indicates more than one sequence of black squares. No indication is given of how many white squares appear in a column, though there is at least one between each sequence of black squares. There may or may not be white squares prior to the first black sequence, and there may or may not be white squares after the last black sequence. For example, column four of the puzzle could begin with a white square followed by two black squares, followed by a white square, followed by three black squares, followed by eight white squares (for a total of fifteen squares). Rows work similarly to columns: the numbers indicated the lengths of sequences of black squares from left to right. When you have filled in the puzzle grid, there will be a simple math problem to solve. The answer to this problem should be written on the answer form you turn in.

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