Mu Alpha Theta National Convention: Denver 2001
Proofs Test – Solutions

1. a. Let $S = a + ar + ar^2 + \cdots$. Multiplying both sides by r yields $rS = ar + ar^2 + ar^3 + \cdots$. Notice that when this quantity is subtracted from S, everything but the first term cancels out. Therefore, $S - rS = a$ or $S = a/(1-r)$.

   b. Again, let $S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$, meaning $rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$. Subtraction produces $S - rS = a - ar^n = a(1-r^n)$. Solving for S, we get $S = a(1-r^n)/(1-r)$.

or

Consider the quantity $s = (1-r^n)/(1-r) = (r^n - 1)/(r-1)$. Using synthetic or long division, we can write s as $r^{n-1} + r^{n-2} + \cdots + r + 1$. Multiply both sides by a and we're done.

2. Let $S_1 = 1 + 2 + 3 + \cdots + n$ while $S_3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$. Our aim is to derive closed formulas for S_1 and S_3. We shall start with S_1. By arranging the terms of the sum in reverse order and adding, we get

$$\begin{aligned} S_1 &= 1 + 2 + 3 + \cdots + (n-1) + n \\ S_1 &= n + (n-1) + (n-2) + \cdots + 2 + 1 \\ 2S_1 &= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ terms}} = n(n+1) \\ S_1 &= \frac{n(n+1)}{2} \end{aligned}$$

Now for S_3. Consider the series $\sum_{i=1}^{n}((i+1)^4 - i^4)$. This is a telescoping sum whose value is $(2^4 - 1^4) + (3^4 - 2^4) + \cdots + (n^4 + (n-1)^4) + ((n+1)^4 - n^4) = (n+1)^4 - 1$. Since $(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$, we have

$$\sum_{i=1}^{n}(4i^3 + 6i^2 + 4i + 1) = (n+1)^4 - 1$$

$$4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = (n+1)^4 - 1$$

$$4\sum_{i=1}^{n} i^3 + 6\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\left(\frac{n(n+1)}{2}\right) + n = (n+1)^4 - 1$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n^2+2n+1)}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

Thus, $S_3 = S_1^2$.

or

Let $\sum_{i=1}^{n} i = n(n+1)/2$ and $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$. We shall use induction to show that these formulas hold for all integers $n \geq 1$. Both are true for the base case $n = 1$ because $\sum_{i=1}^{1} i = 1 = 1(1+1)/2$ and $\sum_{i=1}^{1} i^3 = 1 = 1^2(1+1)^2/4$. Assuming each formula works for integers $n = 1, 2, 3, \ldots, k$ observe that

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}\end{aligned}$$

The two formulas hold for $n = k+1$ so we're done.

3. In $\triangle ABC$, let $BC = a$, $AC = b$, $AB = c$, and D the foot of the altitude to side c. Suppose D splits c into lengths of $AD = x$ and $DB = c - x$. Similar triangles ABC and ACD gives the proportion $AC/AD = AB/AC$ or $b^2 = xc$. From the similarity of $\triangle ABC$ and $\triangle CBD$, we get $BC/BD = AB/BC$ or $a^2 = (c-x)c = c^2 - xc$. Adding these two equations results in $a^2 + b^2 = (c^2 - xc) + xc = c^2$.

or

(*Due to James Garfield, 20th US President*) Consider right triangle ABC with the usual labelling conventions with the right angle at C. Extend BC to point D so that $BD = CA$. Construct segment DE so that DE is perpendicular to CD and $DE = BC$. Draw segments BE and AB. Note that $ACDE$ is a trapezoid whose area is given by $(a+b)(a+b)/2 = (a^2+2ab+b^2)/2$. The area of $ACDE$ can also be found by summing up the areas of triangles ACB, BDE, and EBA. Since $\triangle ABC$ is congruent to $\triangle BDE$ by SAS, $\angle BAC = \angle EBD$ which means $\angle ABE = 90°$. All three triangles have a right angle so their areas are $[ACB] = ab/2$, $[BDE] = ab/2$, and $[ABE] = c^2/2$. Thus, $[ACB] + [BDE] + [ABE] = [ACDE]$ or $ab + c^2/2 = (a^2 + 2ab + b^2)/2$. Simplifying, we get $a^2 + b^2 = c^2$.

4. Let $a_n = a + (n-1)d$. From the given information, we know that a_4 is the geometric mean of a_6 and a_{10} or $(a_4)^2 = a_6 a_{10}$. Substituting the needed quantities, we find that $d = -2a/9$. Thus, $a_n = a + (n-1)(-2a/9) = 11a/9 - 2an/9$ and $S(n)$ is

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \left(\frac{11a}{9} - \frac{2a}{9} i\right) = \frac{11a}{9}(n) - \frac{2a}{9}\left(\frac{n(n+1)}{2}\right) = \frac{an(10-n)}{9}$$

Proving each part naturally follows:

a. By direct application of the formula, $S(10) = 0$.

b. $S(6) + S(12) = 24a/9 - 24a/9 = 0$.

5. The claim is clearly true if any two elements in the set are equal so we can assume the 52 integers are distinct. Work in modulo 100 and consider the set $\{0, 1, 2, 3, \ldots, 99\}$. We partition this collection so numbers that add up to 0 mod 100 are together (in the case where a set only contains one element, that number is added to itself):

$$\{0\}, \{50\}, \{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}$$

Notice that there are 51 compartments. By the Pigeonhole Priciple, if we pick any 52 integers, take their remainder when divided by 100, and place them in the proper compartment, at least two of these compartments will be filled.

6. Make a line segment of length $a+b$ and call this XZ where $a = XV$ and $b = VZ$. Then, construct a unit length YV perpendicular to XZ. Draw the circumcircle of $\triangle XYZ$ and extend YV so it hits the circle at P, making YP a chord of the circumcircle. The length of VP is equal to ab. To see why, apply Power of a Point to the lengths to obtain $(YV)(VP) = (XV)(VZ)$ or $VP = ab$.

or

Construct right triangle EFG (where $\angle EFG$ is the right angle) with $EF = 1$ and $FG = b$. From F, measure outward toward E and create HF such that $HF = a$ and HF lies on the same line as EF. Copy $\angle GEF$ at H and extend the new ray until it intersects the line perpendicular to HF which contains FG; call this intersection point K. Clearly triangles HFK and EFG are similar. Thus, $HF/EF = FK/FG$ or $FK = (HF)(FG)/EF = ab/1 = ab$.

7. a. Suppose that $\sqrt{2}$ can be written as a/b where a and b are relatively prime integers. This implies that $2 = a^2/b^2$ or $2b^2 = a^2$. The left side of this equation is even which means a is even. Letting $a = 2c$, we get $b^2 = 2c^2$ hence, b must be even as well. But this contradicts the assumption that a and b are relatively prime. We conclude that $\sqrt{2}$ is irrational.

or

If $\sqrt{2}$ is rational, there exists at least one integer a such that $a\sqrt{2}$ is an integer. Let S be the set of integers of the form $k\sqrt{2}$, where k is a positive integer. Because this set is nonempty, the Well-Ordering Property guarantees that S has a smallest element, say $s = t\sqrt{2}$. Now let $x = s\sqrt{2} - s$. Notice that x is in S because $x = s\sqrt{2} - s = s\sqrt{2} - t\sqrt{2} = (s-t)\sqrt{2}$ and $s > t$. Also, $x < s$ since $s = t\sqrt{2}$, $s\sqrt{2} = 2t$, and $\sqrt{2} < 2$. This contradicts the choice of s as the smallest member of S and it follows that $\sqrt{2}$ is irrational.

or

Let $f(x) = x^2 - 2$. By the Rational Root Theorem, the only possible rational roots for f are in the set $\{-2, -1, 1, 2\}$. Direct evaluation of these numbers into f shows that none of these values work. Therefore, the roots of f—namely $\sqrt{2}$—must be irrational.

b. We apply induction on n, with the base case $n = 1$ already settled on part a. Now assuming $2^{1/2^n}$ is irrational for integers $n = 1, 2, 3, \ldots, k$, we will show that $2^{1/2^{k+1}}$ is irrational using indirect proof. For the purpose of forming a contradiction, suppose that $2^{1/2^{k+1}} = a/b$, where a and b are integers and $\gcd(a, b) = 1$. Squaring both sides, we get $2^{1/2^k} = a^2/b^2$. The left side of this equation is irrational by the inductive hypothesis while the right side is not, a contradiction. Thus, $2^{1/2^n}$ is irrational for integers $n \geq 1$.

8. Suppose that r and s are rational, where $r < s$. Consider the number $t = r + (s-r)/\sqrt{2}$. Clearly t is in-between r and s and since $\sqrt{2}$ is irrational (by 7a), t is a sum of a rational and an irrational number and is therefore irrational.

9. The given inequality follows from the fact that $(a^3 - 2b^3)^2 \geq 0$ and $ab = 1$.

or

Since the greater side of the inequality is always nonnegative, we can assume without loss of generality that a and b are both positive. Applying the Arithmetic-Geometric Mean Inequality (AM-GM) along with $ab = 1$, we get

$$\frac{a^6 + 4b^6}{2} \geq \sqrt{4a^6b^6} = 2a^3b^3 = 2(ab)^3 = 2$$

Multiply both sides by 2 and we're done.

or

Because $ab = 1$, $b = 1/a$. Substituting this into the expression, we get $a^6 + 4(1/a)^6 = a^6 + 4/a^6 = F(a)$. From calculus we know that the critical values of F give rise to potential extrema. Setting $F' = 0$ produces $6a^5 - 24/a^7 = 0 \rightarrow a = \pm\sqrt[6]{2}$. By the First Derivative Test, F attains a global minimum at both these values. Thus, $F(a) \geq F(\pm\sqrt[6]{2}) = 4$, and the inequality is proven.

10. The roots of $f(x) = x^5 + 1$ are $e^{\pi i/5}$, $e^{3\pi i/5}$, $e^{\pi i}$, $e^{7\pi i/5}$, and $e^{9\pi i/5}$, where $i = \sqrt{-1}$ and $e^{i\theta} = \cos\theta + i\sin\theta$. Since f has no x^4-term, the sum of its roots must equal 0. Consequently, the sum of the real parts of their sum equals zero. Thus

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} + \cos\pi + \cos\frac{7\pi}{5} + \cos\frac{9\pi}{5} = 0$$
$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} + \cos\frac{7\pi}{5} + \cos\frac{9\pi}{5} = 1$$

Let $x = \cos\frac{3\pi}{5}$ and $y = \cos\frac{9\pi}{5}$. Now since $\cos(2\pi - \theta) = \cos\theta$, $x = \cos\frac{3\pi}{5} = \cos\frac{7\pi}{5}$ and $y = \cos\frac{9\pi}{5} = \cos\frac{\pi}{5}$. Plugging all this into the bottom equation above yields $2x + 2y = 1$ so $x + y = 1/2$.

or

Again, let $x = \cos\frac{3\pi}{5}$ and $y = \cos\frac{9\pi}{5}$. By the appropriate sum-to-product formula, $x + y = -2xy$. In addition, since $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, we find that $y = 4x^3 - 3x$. Solving this system, we obtain the solution set $x \in \{0, -1, (1 \pm \sqrt{5})/4\}$. The first two roots are clearly meaningless and since $3\pi/5$ is an obtuse angle, $\cos\frac{3\pi}{5} < 0$ so $x = (1 - \sqrt{5})/4$. Some messy algebra results in $y = (1 + \sqrt{5})/4$, making $x + y = 1/2$.

11. Notice that from a configuration of two distinct triangles, we can arrive at another configuration by permuting the vertices of $\triangle ABC$ and $\triangle DEF$. There are 3! ways of doing this for each triangle. Moreover, there are $6!/6 = 5!$ ways of arranging the points about the circle without restriction. The probability is $(3! \times 3!)/5! = 3/10$.

12. We explicitly construct an integer with the desired properties. Let $n = m^2 + 3m + 3$. Then $m + n + 1 = m^2 + 4m + 4 = (m+2)^2$ and $mn + 1 = m^3 + 3m^2 + 3m + 1 = (m+1)^3$.