# Mu Alpha Theta National Convention: Denver 2001

# Integration Topic Test – Solutions

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1. **(C).** $\displaystyle\int_0^4 x^3 + x + 1\,dx = \frac{x^4}{4} + \frac{x^2}{2} + x\Big|_0^4 = 76$

2. **(A).** Displacement $\displaystyle = \int_a^b v(t)\,dt = \int_0^4 t^3 - 3t^2 + 2t\,dt = \frac{t^4}{4} - t^3 + t^2\Big|_0^4 = 16$

3. **(B).** Let $u = 2a + 1$. Then $du = 2\,da$ so $\int (2a+1)^8\,da = \int u^8/2\,du = u^9/18 + C = (2a+1)^9/18 + C$.

4. **(D).** If $\tan u = x$, then $\sec^2 u\,du = dx$ and the integral becomes $\int x^2(x^2+1)^2\,dx = \int \tan^2 u(\tan^2 u + 1)^2 \sec^2 u\,du = \int \tan^2 u(\sec^2 u)^2 \sec^2 u\,du = \int \tan^2 u \sec^6 u\,du$.

5. **(B).** The graph has roots at $x = \pm 5$ so the area is $\displaystyle\int_5^5 25 - x^2\,dx = \frac{500}{3}$.

6. **(D).** The partitions are $\{[0,1],[1,3],[3,7]\}$. Since $f(x) = x^2$ is strictly increasing when $x > 0$, we pick the right endpoint of each subinterval, resulting in an upper sum of $1(1)^2 + 2(3)^2 + 4(7)^2 = 215$.

7. **(A).** Notice the integrand is simply the Product Rule. Thus, $\int_1^3 a'(t)b(t) + a(t)b'(t)\,dt = a(3)b(3) - a(1)b(1) = (7)(15) - (5)(3) = 90$.

8. **(A).** The integrand is none other than the Chain Rule. Therefore, $\int_0^1 a'(b(t))b'(t)\,dt = a(b(1)) - a(b(0)) = a(3) - a(1) = 7 - 5 = 2$.

9. **(C).** Let $u = t/3$. Then $du = dt/3$ and the integrand transforms into the Quotient Rule

$$\int_0^3 \frac{a'(t/3)b(t/3) - b'(t/3)a(t/3)}{(b(t/3))^2}\,dt = 3\int_0^1 \frac{a'(u)b(u) - b'(u)a(u)}{(b(u))^2}\,du$$
$$= 3\left(\frac{a(1)}{b(1)} - \frac{a(0)}{b(0)}\right) = -7$$

10. **(D).** Given a rate of change function $r(t)$, the total change over a given time interval is the integral of $r(t)$ over the interval. Therefore, the total number of girls in Tom's house after 81 minutes is $\int_0^{81} 20 + 4\sqrt{t}\,dt = 3564$.

11. **(A).** Recall that if $f(x)$ is an odd function, $\int_a^a f(x)\,dx = 0$. Thus, the integers chosen must be $\{-5,5\}$, $\{-4,4\}$, $\{-3,3\}$, $\{-2,2\}$, or $\{-1,1\}$. There are $\binom{11}{2} = 55$ ways of choosing the integers without restriction so the probability is $5/55 = 1/11$.

12. **(C)**. When $-2 \leq x < -1$, $x+1 < 0$ and if $-1 \leq x < 1$, $x+1 \geq 0$ so we can break up the integral as $\int_{-2}^{1} |x+1|\,dx = \int_{-2}^{-1} -x-1\,dx + \int_{-1}^{1} x+1\,dx = 1/2 + 2 = 5/2$

13. **(A)**. Average Value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(n)\,dn = \frac{1}{2}\int_0^2 n^2\,dn = \frac{4}{3}$

14. **(C)**. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$. Thus

$$\int_1^2 n(z)\,dz = \frac{1}{2}\left(\int_{-2}^{2} n(z)\,dz - \int_{-1}^{1} n(z)\,dz\right) = \frac{1}{2}(4-(-6)) = 5$$

15. **(D)**. By setting $y$-values equal to each other, we see that the graphs intersect at $x = -2, 0, \text{and}\, 2$. When $-2 \leq x \leq 0$, $3x^3 - x^2 - 10x \geq 2x - x^2$ and when $0 \leq x \leq 2$, $2x - x^2 \geq 3x^3 - x^2 - 10x$. Thus, the total area is given by

$$\int_{-2}^{0} (3x^3 - x^2 - 10x) - (2x - x^2)\,dx + \int_0^2 (2x - x^2) - (3x^3 - x^2 - 10x)\,dx$$

which yields a value of $12 + 12 = 24$.

16. **(D)**. Completing the square, we see that $x^2+12x+45 = (x+6)^2+9$. By the arctangent integration formula, the answer is

$$\int \frac{dx}{x^2+12x+45} = \int \frac{dx}{(x+6)^2+3^2} = \frac{1}{3}\arctan\frac{x+6}{3} + C$$

17. **(D)**. The area of the lamina is $\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx = 2\sqrt{3}/3$. Therefore, its mass is $(18)(2\sqrt{3}/3) = 12\sqrt{3}$ kilograms.

18. **(B)**. The Mean Value Theorem for Integrals states that the a function will equal its average value over an interval. Thus, we have

$$u^2 + 1 = \frac{1}{3}\int_2^5 u^2 + 1\,du = 14$$

The only solution on the given interval is $u = \sqrt{13}$.

19. **(B)**. It's easy to see from a graph that the limits of integration are the solutions to $10y - 16 - y^2 = 0$ or $\{2, 8\}$. We also know that $a < b$, or else the value of the integral will be negative. Thus, $(a, b) = (2, 8)$.

20. **(B)**. $\displaystyle\int_e^{e^2} \frac{1}{x} + \frac{1}{x^2}\,dx = \ln|x| - \frac{1}{x}\Big|_e^{e^2} = \frac{e^2+e-1}{e^2}$

21. **(A)**. By the Second Fundamental Theorem, $F'(t) = (3t^2)\sin(t^3)^2 = 3t^2 \sin t^6$.

22. **(B)**. $\int \frac{r^2+3r^3-r^5}{r\sqrt{r}}\,dr = \int r^{1/2}+3r^{3/2}-r^{7/2}\,dr = \frac{2}{3}r^{3/2}+\frac{6}{5}r^{5/2}-\frac{2}{9}r^{9/2}+C$

23. **(D)**. We have $\int_0^\pi x+\sin x\,dx = \pi^2/2+2$ and $\int_\pi^{2\pi} x+\sin x\,dx = 3\pi^2/2-2$ so the ratio of the larger area to the smaller is $(3\pi^2/2-2)/(\pi^2/2+2) = (3\pi^2-4)/(\pi^2+4)$.

24. **(A)**. By the Shell Method, the volume is $2\pi\int_0^1 x(x^3-x^2+x+1)\,dx = \frac{47\pi}{30}$.

25. **(C)**. We use the area interpretation of the integral. Notice that the graph of $f(x) = \lfloor x\rfloor$ creates a staircase of rectangles each having a width of 1 and a height equal to $f(x)$. There are 10 rectangles on the interval $0 \le x \le 10$, giving us a total area of $1(0+1+2+\cdots+9) = 45$.

26. **(D)**. From the double-angle formula, $\sin 2x = 2\sin x\cos x$ or $\sin x\cos x = (\sin 2x)/2$. Thus, $\int \sin x\cos x\,dx = \int(\sin 2x)/2\,dx = -(\cos 2x)/4+C$.

27. **(C)**. We actually want the $x^4$-term of $(2x-1)^{12}$ since integration of a polynomial involves adding 1 to the exponent. This term is $\binom{12}{4}(2x)^4(-1)^8 = 7920x^4$. Integration produces $\int 7920x^4\,dx = 1584x^5+C$ so the desired coefficient is 1584.

28. **(D)**. If the slope of the normal line at $(x,y)$ is $x^2y$, then the slope of the tangent line—better known as $dy/dx$—is equal to $-1/(x^2y)$. Thus, $dy/dx = -1/(x^2y)$. Using separation of variables, we get

$$y\,dy = -\frac{1}{x^2}\,dx \rightarrow \int y\,dy = \int -\frac{1}{x^2}\,dx \rightarrow \frac{y^2}{2} = \frac{1}{x}+C$$

Letting $x=-2$ and $y=1$, we get $C = 1/2-(1/(-2)) = 1$. Thus, $y^2/2 = 1/x+1$ or $y^2 = (2x+2)/x$.

29. **(A)**. We apply Integration by Parts twice, always letting $u$ equal the polynomial part and $dv$ the trigonometric function.

$$\begin{aligned}\int \theta^2\cos\theta\,d\theta &= \theta^2\sin\theta - 2\int\theta\sin\theta\,d\theta\\ &= \theta^2\sin\theta - 2\left(-\theta\cos\theta + \int\cos\theta\,d\theta\right)\\ &= \theta^2\sin\theta + 2\theta\cos\theta - 2\sin\theta + C\end{aligned}$$

30. **(B)**. Evaluating the integral, we obtain the recursive sequence $a_{n+1} = 3a_n - 2a_{n-1}$. The next three terms of this sequence are $a_2 = 3a_1-2a_0 = 3$, $a_3 = 3a_2-2a_1 = 7$, and $a_4 = 3a_3-2a_2 = 15$, so we might conjecture that $a_n = 2^n-1$, which is easily shown to be correct using induction. The sum we seek is

$$\begin{aligned}\sum_{n=1}^{2001}(2^n-1) &= (2+2^2+2^3+\cdots+2^{2001}) - \underbrace{(1+1+1+\cdots+1)}_{2001\text{ ones}}\\ &= \frac{2(2^{2001}-1)}{2-1} - 2001 = 2^{2002}-2003\end{aligned}$$

31. **(D)**. On the interval $0 \le x \le 2\pi$, the graphs of the sine and cosine intersect at $x = \pi/4$ and $x = 5\pi/4$. The shared area is therefore equal to $\int_{\pi/4}^{5\pi/4} \sin x - \cos x \, dx = 2\sqrt{2}$.

32. **(A)**. $\displaystyle\int 10^x \, dx = \frac{10^x}{\ln 10} + C$

33. **(A)**. By the Washer Method, the volume is $\displaystyle\pi \int_4^6 (2x)^2 - (x)^2 \, dx = \pi \int_4^6 3x^2 \, dx = 152\pi$.

34. **(D)**. Area $\displaystyle= \int_{\sqrt{3}/3}^{\sqrt{3}} \frac{x^3+x+1}{x^2+1} \, dx = \int_{\sqrt{3}/3}^{\sqrt{3}} x + \frac{1}{x^2+1} \, dx = \frac{x^2}{2} + \arctan x \Big|_{\sqrt{3}/3}^{\sqrt{3}} = \frac{4}{3} + \frac{\pi}{6}$

35. **(C)**. The value of the integral is $\int_0^x 5n^4 - 1 \, dn = x^5 - x$ which, by Fermat's Little Theorem, is divisible by 5 for all integers $x$. Therefore, the desired sum is simply that of the first 67 natural numbers or $(67)(68)/2 = 2278$.

36. **(B)**. By the Washer Method, the volume is $\pi \int_0^1 (kx^2+1)^2 - 1^2 \, dx = 56\pi/5$ or after some work, $2k/3 + k^2/5 = 56/5$. The solution set is $k \in \{6, -28/3\}$ so $k = 6$.

37. **(C)**. Plug $y = 0$ into the identity to obtain $3f(x) + 2f(x) = 5x^2$ or $f(x) = x^2$.

38. **(C)**. Given a regular octagon with side length $s$, its area is given by $A = 2s^2(1+\sqrt{2})$. In this case, the graph of the function gives the side length of the octagon and it follows that the volume is $\int_0^4 2(\sqrt{x})^2(1+\sqrt{2}) \, dx = 16 + 16\sqrt{2}$.

39. **(B)**. Let $I = \int_0^{\pi/2} (\sin x)/(\sin x + \cos x) \, dx$. Now consider the integral

$$J = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} \, dx$$

Notice that $I = J$ since $\cos x$ attains the same values as $\sin x$ on the interval $[0, \pi/2]$, only in reverse order (verify this by drawing the respective graphs). Adding the two integrals, we get

$$I + J = 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} \, dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

Thus, $I = \pi/4$.

40. **(D)**. By the Shell Method, the volume is $\displaystyle 2\pi \int_3^5 x(3x - 2x) \, dx = 2\pi \int_3^5 x^2 \, dx = \frac{196\pi}{3}$.