1. **The easiest way to solve this problem is to find the total possible values of m because for each m there is exactly one possible n to pair with. The possible values of m consist of the total number of possible factors (positive and negative) of 840. The total number of positive factors of a number can be derived from that number's prime factorization. Add one to each of the exponents in the prime factorization (the total number of choices for the exponent of that prime in a factor) and multiply those numbers together (they are chosen independently from one another to identify each factor). The prime factorization of 840 has four primes (2, 3, 5, 7) with exponents of 3, 1, 1, and 1 respectively. So there are (3+1)(1+1)(1+1)(1 + 1) = 32 possible positive values for m, and also 32 negative values. For each m, there is exactly one n. There are 64 different ordered pairs.**

2. **15,625 =** 5^6**.** $\sum_{n=0}^{5} 5^n$ **= (**5^6**-1)/(5-1) = 3,906.**

3. **The trick is to find the number of powers of 10 which divide 15,634!. This means finding the exponents of 2 and 5 in the prime factorization of 15,634!. The exponent of 2 will clearly be greater than the exponent of 5, so we only need find the exponent of 5. 15,634/5 = 3,126 with a remainder. 3,126/5 = 625 with a remainder. 625/5 = 125. 125/5 = 25. 25/5 = 5 and 5/5 = 1. That means that there are 3,126 + 625 + 125 + 25 + 5 + 1 = 3,907 times in which 5 is included in the prime factorization of 15,634!. Thus 3,907 is the answer.**

4. **Using the same method of factor counting as in problem #1, we can arrange 32 as the product of exponents (+1) in several ways. 32 = (4)(2)(2)(2) = (4)(4)(2), etc. We can make the problem easier by noting that we can produce a smaller number with four factors using only powers of 3 than with a single factor of 3 and a prime greater than 3 x 3 (for instance, 27 is less than 33 or 39). Noting such relationships we can see that 32 = (4)(2)(2)(2) produces the smallest possible integer using the smallest odd primes (3, 5, 7, and 11). (3)(3)(3)(5)(7)(11) = 10,395.**

5. **There is a theorem that I have heard called the "Chicken Mcnugget Theorem" which gives a solution for a diophantine equation with only two relatively prime variables (a and b) instead of three. The formula is fairly easy to derive and shows that d = ab – a – b. I leave it as an exercise to the solvers to derive the formula for more variables.**

6. **91 = 7 x 13. 111 = 3 x 37. 297 = 3 x 3 x 3 x 11. The LCM is thus 3 x 3 x 3 x 7 x 11 x 13 x 37 = 999,999.**

7. **82,861 = 41 x 43 x 47. 41 + 43 + 47 = 131.**

8. 432_8 **= 4(64) + 3(8) +2 = 282 =** n^2**+ 8n + 9. Thus** n^2**+ 8n – 273 = 0. Solutions for n are 13 and –21. Discard the negative solution.**

9. There is a simple theorem in modular arithmetic that says that when we are looking for a set of number s with the same congruence in two different mods, then we are looking for the set of numbers that has that same congruence in the LCM of the two previous mods. The LCM of 3 and 11 is 33. We are looking for the smallest prime with a congruence of 1 (mod 33). A quick search reveals 67 as the answer.

10. Rewriting the Fibonacci numbers in (mod 3) reveals a pattern which repeats in cycles of 4 with only one of the numbers in that cycle being congruent to 0 (mod 3). Thus exactly 100 of the first 400 Fibonacci's are multiples of 3.

11. 280 is the only one of the answer choices which leaves a remainder of 7 when divided by 13.

12. Call the number AB where A is the tens digit and B is the units digit.

BA – AB = 10(B - A) + (A - B) = 9(B - A) = 27. Thus B – A = 3.

13. A quick way of solving this problem is to note that 121_8 = (8 + 1)(8 + 1) = (7 + 2)(7 + 2) = 144_7. Base numbers still obey regular algebraic manipulation.

14. 840 = 2^3 x 3 x 5 x 7. Consider that the sum of all of the factors can be determined by:

$(2^3 + 2^2 + 2^1 + 1)(3^1 + 1)(5^1 + 1)(7^1 + 1)$ because each and every factor is represented one and only once as one of the products of a power of each of the prime numbers in 840's prime factorization. Also notice that $(2^3 + 2^2 + 2^1 + 1)(2 - 1) = (2^4 - 1)$. So a more compact formula can be derived to find the sums of factors of ANY integer. [Try to derive this formula completely as an exercise.]

The sum of all of its positive factors is thus

$[(2^4 -1)(3^2 -1)(5^2 -1)(7^2 -1)]/[(2 - 1)(3 - 1)(5 - 1)(7 - 1)] = 2880$.

15. The sum, A, need only be calculated up to the point at which all subsequent terms are multiples of 144. Thus only the first 5 terms need be calculated. The sum of those terms is 1 + 2 + 6 + 24 + 120 = 153. The remainder when 153 is divided by 144 is 9.

16. First not that B ≡ -1 (mod 3) and B ≡ -1 (mod 8) => B ≡ -1 (mod 24). Thus B ≡ -1 (mod 12) => B ≡ 11 (mod 12).

17. The greatest integer less than the cube-root of 100,000 is 46. The sum of the first 46 positive perfect cubes is the square of the 46th triangular number. (46)(47)/2 = 1,081. (1,081)(1,081) = 1,168,561.

18. $(15^3)(10^5) = (2^5)(3^3)(5^8)$. **Divide both this number and 128 by 32. Now take the remaining portion of the large number and find its remainder when divided by 4 (which is 128/32):** $(3^3)(5^8) \equiv (-1)^3(1^8) \pmod 4 \equiv -1 \pmod 4 \equiv 3 \pmod 4$. **Multiplying both sides back by 32 tells the solver that the original number is congruent to 96 (mod 128).**

19. $42a + 3(40a) \equiv 0 + 3(1) \pmod 7$. **Thus** $162a \equiv 3 \pmod 7$.

20. $5A55B \equiv 0 \pmod{72}$ => $5A55B \equiv 0 \pmod 8$ **and** $5A55B \equiv 0 \pmod 9$.

$5A55B \equiv 0 \pmod 8$ => $55B \equiv 0 \pmod 8$ => $B + 6 \equiv 0 \pmod 8$. $B = 2$.

$5A55B \equiv 0 \pmod 9$ => $A + B + 15 \equiv 0 \pmod 9$ => $A + 17 \equiv 0 \pmod 9$. $A = 1$.

21. The difference between their ages must be a multiple of the LCM of Katie's ages on those 7 birthdays. The smallest that LCM could be is 420. On the last birthday Bart must have been 427.

22. N is 2 less than a multiple of 3, 5, 7, and 9. The LCM of 3, 5, 7, and 9 is 315. 315 – 2 = 313.

23. $7^4 = 2{,}401 \equiv 400 + 1 \pmod{1{,}000}$. **Take that result to the fifth power:**

$7^{20} \equiv (400 + 1)^5 \pmod{1{,}000} \equiv 5(400) + 1 \pmod{1{,}000} \equiv 1 \pmod{1{,}000}$.

Thus $7^{404} \equiv 7^{20(20)}\, 7^4 \pmod{1{,}000} \equiv (1)(401) \pmod{1{,}000} \equiv 401 \pmod{1{,}000}$.

Hence the hundreds digit of 7^{404} **is 4.**

24. We are hunting for integers, N, such that $N \equiv 5 \pmod{16}$, $N \equiv 6 \pmod{25}$, **and** $N \equiv 7 \pmod{36}$. **From the first of these relations, we know that** $N \equiv 1 \pmod 4$**, but that contradicts the third relation which shows that** $N \equiv 3 \pmod 4$. **Thus there are no such integers.**

25. $11x \equiv 65 \pmod{67}$ => $11x \equiv 65 + 67 \pmod{67}$ => $11x \equiv 132 \pmod{67}$ => $x \equiv 12 \pmod{67}$. **All of the answers except for 6,567 are congruent to 12 (mod 67).**

26. We are looking for natural numbers that are perfect squares. First, we sum the first 51 positive perfect squares to get (51)(52)(103)/6 = 45,526. We then subtract 1 to get 45,525.

27. Given $x \equiv 2 \pmod 4$, we can say x = 4a –2 for some positive integer, a. Then from the second equation, $4a - 2 \equiv 3 \pmod 9$ => $4a \equiv 5 \pmod 9$ => $a \equiv 8 \pmod 9$ and thus we can say a = 9b – 1 for some positive integer, b. Thus x = 36b – 6. Finally, from the last equation, $36b - 6 \equiv 5 \pmod{25}$ => $36b \equiv 11 \pmod{25}$ => $b \equiv 1 \pmod{25}$. Thus we can say that b = 25c – 24 for some positive integer, c. Thus x = 900c – 870. 930 is the second smallest positive solution.

28. An integer expressed in a base, B, is a multiple of (B – 1) if and only if the sum of the integer's digits is a multiple of (B – 1). This can be easily proven by induction (or other means) and is left as an exercise for the students. Now, we need only look at numbers with digit sums of 4 (8, 12, etc. would be far too large) in base 5. The second smallest of these is $10{,}111_5$ = 656.

29. The product can be written as x^2 - 1 and factored into (x + 1)(x – 1). If the only prime factors are between 60 and 80, they must have a difference of [(x + 1) – (x – 1)] = 2. The only twin primes in that range are 71 and 73. 73 is the larger.

30. Notice that AAAA can be factored into (AA)(101). Also, AA + 2 = 101. Q is thus 101.

31. This problem requires separating the cases of p =2 and p =3 from other cases because they are factors of 24. They clearly have their own solutions for b. For p > 3 we will evaluate: We can first write p as (2m + 1) and then evaluate $p^2 \pmod 8$.

$p^2 = 4m^2 + 4m + 1 = 4m(m + 1) + 1$. Either m or (m + 1) is even, thus

$4m(m+1) + 1 \equiv 1 \pmod 8$. Also, such p are either congruent to 1 or –1 (mod 3) and thus p^2 is congruent to 1 (mod 3). p^2 is thus also congruent to 1 (mod 24). This is true for all primes which are not 2 or 3 and so there are exactly 3 possible values for b.

32. The product of eight consecutive triangular numbers can be written as the product of eight consecutive integers (starting with n) and then another eight consecutive integers (starting with n + 1), divided by 2 to the eighth power. It is easy to see that out of any eight consecutive integers, there are 4 even integers, 2 multiples of 4, and 1 multiple of 8. Overall, the product of the eight consecutive triangular numbers must therefore be a multiple of 64.

33. The sum of the first n counting numbers is always the square of the nth triangular number. The only even number that is not composite is 2. Thus 49 of the first 50 even counting numbers are stupid political candidates.

34. (A, B, C, D) = (A, B, 2B, 8B). 8B is a multiple of 9 meaning that B is a multiple of 9. Let B = 9x, x is an integer. Then A + B + C + D = A + 99x = 100. Since x is an integer, x can only be 1 and thus A = 1.

35. By Fermat's Little Theorem we know that $5^{11-1} \equiv$ 1 (mod 11). From this we can determine that $5^{(10)(20)+1} \equiv (1^{20})(5)$ (mod 11) $\equiv$ 5 (mod 11).

36. The number in question is at least 2^{31} and less than 2^{32}. There are a couple of ways to do the problem from here. Some students may recognize that log2 $\cong$ .301. (31)(.301) = 9.331 and (32)(.301) = 9.632. Thus both 2^{31} and 2^{32} are ten digit numbers (logs between 9 and 10). It could also be noted that $2^{10} \cong 10^3$ and thus $2^{31} \cong 2(10^9)$ and is a ten digit number (similarly for 2^{32}). Obviously all numbers between 2^{31} and 2^{32} are also ten digit numbers.

37. Solving this problem involves a degree of deduction taking several factors into consideration. We can rule out even values of *N*. We can also note that phi(10) = 4 and phi(100) = 40. This will help limit our search as we know that the units digit of 3^N repeats in a 4-cycle and the last pair of digits repeats in (at most) a 40-cycle. In fact, noting that 3 x 3 x 3 x 3 = 81 = (80 + 1), we can see by binomial expansion that taking 81 to the fifth power produces a number with a units digit of 1 and a tens digit of 0. Thus 3^N repeats its last two digits in a 20-cycle. Now we must simply look for where $3^N \equiv 0$ (mod 20) and adjust *N* by adding/subtracting multiples of 20. We thus need only check the first 20 positive integers (and only the 10 odd ones of those).

We can rule out most of these by comparing the 4-cycle of units digits. If $N \equiv 1$ (mod 4), then the units digit of 3^N will be 3. If $N \equiv 3$ (mod 4), the units digit will be 7. The only *N* that need be tested are thus 7 and 13. $3^N \equiv N$ (mod 20) for 7, but not 13. The tens digit of 3^7 is 8, thus 87 is the only solution such that $3^N \equiv N$ (mod 100).

38. Given K $\equiv$ 1 (mod3), we can equate K = 3a – 2 for some positive integer, a. Substituting for K into K $\equiv$ 3 (mod 5) yields the relationship 3a – 2 $\equiv$ 3 (mod 5) => 3a $\equiv$ 0 (mod 5) => a $\equiv$ 0 (mod 5) and so we can equate a = 5b for some positive integer b and thus K = 15b – 2. Substituting for K into K $\equiv$ 7 (mod 11) yields 15b – 2 $\equiv$ 7 (mod 11) => 15b $\equiv$ 9 (mod 11) => 5b $\equiv$ 3 (mod 11) => 5b $\equiv$ 25 (mod 11) => b $\equiv$ 5 (mod 11) and so we can relate b = 11c – 6 for some positive integer, c. Thus K = 165c – 92.

The sum is equivalent to $\sum_{n=1}^{20}(165n-92)$ **=** $(\sum_{n=1}^{20}165n)$ **– (20)(92) = 165**$(\sum_{n=1}^{20}n)$ **– 1,840**

165$(\sum_{n=1}^{20}n)$ **– 1,840 = 165(20)(21)/2 – 560 = 34,650 – 1,840 = 32,810.**

39. The Fibonacci numbers will always be cyclical in any mod because a term is defined by its predecessors and there are a limited number of possible combinations for a pair of predecessors which would then produce the same cyclical pattern each time that pair occurs. The trick is just to write down the modular residues of the Fibonacci numbers (mod 9) until the cycle is found: 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 0, 1, 1, etc. The 1st pair of terms (1, 1) reappeared as the 25$^{\text{th}}$ **pair. The cycle is thus a 24-cycle, thus m = 24.**

40. This problem is much easier given a knowledge of the arithmetic mean-geometric mean (A.M.-G.M) inequality. For any two positive numbers a and b, their A.M. is greater than or equal to their G.M. Given a fixed G.M., the A.M. is smaller when the difference between a and b is smaller (this is left as an exercise for the students to prove). So, the solution to this problem involves finding three numbers that are relatively prime and considering whether or not there is a way to reduce the A.M. of any pair of them (given that they HAVE a G.M. already). The numbers turn out to be 5, 9, and 16. The sum is 30.