SOLUTIONS---Limits & Derivatives

FAMAT State Convention 2002

1. B-- $y = x^{3/2} \Rightarrow y' = \frac{3}{2}x^{1/2} \Rightarrow y'' = \frac{3}{4}x^{-1/2} \Rightarrow y''' = \frac{-3}{8}x^{-3/2} \Rightarrow y'''(4) = \frac{-3}{64}$

2. D--This is the def of deriv. The deriv of sec x is secx tanx. Eval. at x = $\frac{\pi}{3}$ gives the answer.

3. A-- $\lim_{x\to 0}\frac{-\cos x + 1 - \sin x}{x} = \lim_{x\to 0}\frac{(1-\cos x)}{x} - \lim_{x\to 0}\frac{\sin x}{x} = 0 - 1 = -1.$

4. A-- $g'(x) = -\csc x \cot x - (-\csc^2 x) \Rightarrow g'\left(\frac{\pi}{6}\right) = -2\sqrt{3} + 4$

5. B-- $\frac{dy}{dx} = \frac{4x}{y}$. Separation of variables gives $ydy = 4xdx$. Integrating both sides gives $\frac{y^2}{2} = \frac{4x^2}{2} + C$. Plugging in $y = 4$ and $x = -1$ gives C =6. At y = 6, x = $\pm\sqrt{6}$ and $-\sqrt{6}$ is a choice.

6. E-- $y' = 2\cos(3x)[-\sin x] \bullet 3 = -6\cos(3x)\sin(3x) \Rightarrow y'\left(\frac{\pi}{4}\right) = -6\left(\frac{-\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = 3.$

7. B--The derivative of the integral is $\sin\left(\frac{x}{2}\right)\bullet\frac{d\left(\frac{x}{2}\right)}{dx} = \frac{1}{2}\sin\left(\frac{x}{2}\right)$.

8. B-- $f'(x) = 20x^3 - 6x^2 \Rightarrow f'(-1) = -26$ and $g'(10) = \frac{1}{f'(g(10))} = \frac{1}{f'(-1)} = \frac{-1}{26}$.

9. D-- $\lim_{x\to 0}\frac{\sin^2(3x)}{x^2} = \left(3\lim_{x\to 0}\frac{\sin(3x)}{3x}\right)^2 = (3\bullet 1)^2 = 9.$

10. A--The Intermediate Value Theorem guarantees that $f'(x) = 0$ somewhere btwn x =1 and x = 2, as $f'(x)$ changes sign btwn those points. It does not necessarily change signs btwn x =0 and x = 1 or btwn x = 2 and x = 3.

11. E--The function is increasing when the derivative is positive.

12. B--As $f(x)$ approaches $\pm\infty$ as x approaches –1, then III is certainly true. Since $f'(x) = \frac{1}{(x+1)^2}$, which is never = 0, it has no local maxima. The 2nd derivative is $\frac{-2}{(x+1)^3}$, which is likewise never = 0, so there is no point of inflection.

13. B--The function has a horizontal tangent line when the derivative = 0. Differentiation gives $3x^2 - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$. Thus, a horizontal tangent occurs at $\frac{3x^2 - 3y}{3x - 2y} = 0$, or when $x^2 = y$. Plugging in x^2 for y gives $x^4 - 2x^3 = 0$. And, the tangent line is horizontal at (2, 4).

14. C--The derivative is $\tan x + x\sec^2 x$. Plugging in $x = \frac{\pi}{4}$ gives the answer.

15. D--As air is being added to the balloon, the radius is certainly positive and increasing. However, as the balloon gets larger, the volume of air needed to increase the radius by one centimeter goes up, since air is being added at a constant rate, the rate at which the radius increase goes down. Thus, $r''(t)$ is negative.

16. A--The derivative of $f(x)$ is $e^{x^2-1}(-\sin(x-1)+2x\cos(x-1))$. Plug in x = 1 to get the answer.

17. A--The 2nd derivative of the function is $\dfrac{-(2x^2+4x)}{(x^2+2x+2)^2}$, which = 0 when x = 0 and x = -2.

18. C--If the function is continuous, then plug 1 into the top and bottom pieces and you should get the same answer. This gives us $a-6=b+4$. If the function is differentiable, then plug 1 into the top and bottom of the function and you should get the same answer. This gives $3a-6=2b$. Solving these 2 equations gives $a=-14$.

19. E--$\dfrac{f(b)-f(a)}{b-a}=\dfrac{f(5)-f(2)}{3}=16$. Set this = to $f'(c)=6c-5$ gives $6c-5=16$ or $c=\dfrac{7}{2}$.

20. D--Rewriting the equation as $f(x)=\left(1+x^{1/2}\right)^{1/2}\Rightarrow f'(x)=\dfrac{1}{2}\left(1+x^{1/2}\right)^{1/2}\left(\dfrac{1}{2}x^{-1/2}\right)$ or answer D

21. D--Implicit differentiation gives $0=x\dfrac{dy}{dx}+y-\left[\left(x\dfrac{dy}{dx}+y\right)e^{xy}\right]\Rightarrow\dfrac{dy}{dx}=\dfrac{ye^{xy}-y}{x-xe^{xy}}=\dfrac{-y}{x}$.

22. C--Remember, $\dfrac{dy}{dt}=\dfrac{dy}{dx}\bullet\dfrac{dx}{dt}=(10x+4)\left(\dfrac{1}{t}\right)=\dfrac{10\ln t+4}{t}$.

23. C--Let $y=\sin x$, then this gives $\lim\limits_{y\to 0}4\left(\dfrac{\sin y}{y}\right)=4\bullet 1=4$.

24. E--$\dfrac{dy}{dx}=3(x^2-1)^2(2x)=0\ @\ x=0,\pm 1$. Evaluating the 2nd derivative at these points, gives the absolute minimum at x = 0.

25. B--$x\dfrac{dy}{dx}+y+2y\dfrac{dy}{dx}=0\Rightarrow\dfrac{dy}{dx}=\dfrac{-y}{x+2y}\ @(1,1)=\dfrac{-1}{3}$. Differentiating again gives $\dfrac{d^2y}{dx^2}=\dfrac{-(x+2y)\dfrac{dy}{dx}+y(1+2\dfrac{dy}{dx})}{(x+2y)^2}\ @(1,1)$ and using $\dfrac{dy}{dx}=\dfrac{-1}{3}$ gives $\dfrac{4}{27}$.

26. D--Using L'Hopital's Rule gives $\lim\limits_{x\to 1}\dfrac{1/x}{\pi\sec^2(\pi x)}$. Taking the limit gives $\dfrac{1}{\pi}$.

27. E--$v(t)=6t^2-24t+18=0\ @\ t=1,3$. checking in the 2nd derivative, the acceleration is only 0 at t = 2, so the particle does change direction at both t =1 and t = 3.

28. D-- The prob.is $y=e^{\ln^2 x}$, then $\dfrac{dy}{dx}=e^{\ln^2 x}(2\ln x)\left(\dfrac{1}{x}\right)=(e^{\ln x})^{\ln x}\bullet(2\ln x)\left(\dfrac{1}{x}\right)=x^{\ln x}\bullet(2\ln x)\left(\dfrac{1}{x}\right)$.

29. D--This is a geometric series with a = 1 and r = 1/e. Sum = $\dfrac{a}{1-r}$

30. B—Separating variables gives $\frac{y\,dy}{4+y^2} = x^2dx$. Integrating both sides yields $\ln(4+y^2) = \frac{2}{3}x^3 + C$ and $y^2 = Ce^{\frac{2x^3}{3}} - 4$. Using y = 12 when x = 0, gives C = 148.