2002 National Mu Alpha Theta Convention
Theta Division–Ciphering Solutions

1. **(3)** Since $12^2 = 2^4 \cdot 3^2$ is not divisible by 27 and $12^3 = 2^6 \cdot 3^3$ is, our answer is $n = 3$.

2. **(19600)** Squaring both sides gives us $\sqrt{x} + 4 = 144$. Subtracting 4 and squaring again gives $x = 19600$.

3. **(60)** The number of distinct permutations of the letters in GEORGE is

$$\frac{6!}{2! \cdot 2!} = 180.$$

The R will be between the G's in exactly 1/3 of these (the other two possibilities being the R before both G's and after both G's), so there are 60 distinct permutations that satisfy the problem.

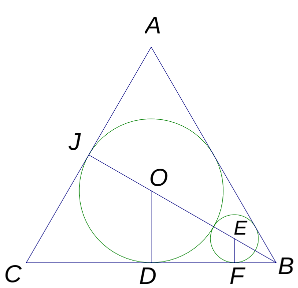
4. **(15/17)** Multiplying both sides by $2x^2 + 11x - 21 = (2x - 3)(x + 7)$ gives us

$$x + 5 = A(x + 7) + B(2x - 3).$$

Letting $x = -7$ gives us $-2 = -17B$ or $B = 2/17$. Letting $x = 3/2$ gives us $13/2 = A(17/2)$ or $A = 13/17$. Thus, $A + B = 15/17$.

5. **(9/91)** There are $5^3 = 125$ ways the dice can be rolled and have no 1's come up, so there are $6^3 - 5^3 = 91$ ways the dice can be rolled and have at least one 1 come up. Of those, there are 9 in which the sum is 6: three ways the dice can come up $1 - 1 - 4$ and six ways $1 - 2 - 3$. Thus, the desired probability is 9/91.

6. **($\pi/3$)** In the diagram, let $OD = R$ and $EF = r$ be radii of the two circles. Since $\angle EBC = 30°$, $OB = 2OD = 2R$ and $EB = 2EF = 2r$. Since $OB = OE + EB$, we have $2R = (R + r) + (2r)$, or $R = 3r$. From right triangle CJB, we have $JB = BC\sqrt{3}/2 = 3\sqrt{3}$. Since O is the centroid of ABC, $OJ = JB/3 = \sqrt{3}$. Thus, $r = \sqrt{3}/3$ and the area of the small circle is $\pi/3$.



7. **(2)** Since

$$\sum_{i=0}^{\infty} \left(\frac{a}{b}\right)^i = \frac{1}{1 - \frac{a}{b}}$$

if $a < b$ and diverges otherwise, our problem reduces to finding pairs (a, b) such that

$$\begin{aligned} \frac{1}{1 - \frac{a}{b}} &=\leq 2 \\ \therefore\ 1 &= 2 - \frac{2a}{b} \\ \therefore\ \frac{a}{b} &= \frac{1}{2} \end{aligned}$$

The only such pairs with $0 < a < b < 6$ are $(1, 2)$ and $(2, 4)$, for a total of 2.

8. **(16π)** Rearranging, our ellipse is

$$9(x^2 + 10x + 25) + 4(y^2 - 2y + 1) = -85 + 225 + 4,$$

or

$$\frac{(x+5)^2}{16} + \frac{(y-1)^2}{36} = 1.$$

The largest circle that can fit inside an ellipse has a diameter equal in length to the minor axis of the ellipse. Thus, the diameter of the largest circle that can be inscribed in the ellipse in question is 8, and the desired area is 16π.

9. **(71)** Since $3k \equiv 4 \pmod 7$, then $3k = 4 + 7n$ for some pair of integers (k, n). When $n = 1$, k is not an integer, but when $n = 2$, $k = 6$. Thus, we set $n = n' + 2$, so our equation is $3k = 18 + 7n'$, or

$$k = 6 + \frac{7n'}{3}.$$

Hence, for every n' which is divisible by 3, we have a solution to our original equation. The solutions which are less than 500 are $k = 6, 13, 20, \ldots, 496 = 7 * 70 + 6$. Thus, there are 71 solutions less than 500.

10. **(92)** Since $|AB| = |A||B|$, where $|A|$ is the determinant of matrix A, we have

$$|AB| = |A||B| = (2 + 21)(0 + 4) = 92.$$

11. **($6 \pm 3\sqrt{3}$)** Multiplying both sides by $3z$ yields $z^2 + 9 = 12z$. Thus, $z^2 - 12z + 9 = 0$ and our solutions are

$$z = \frac{12 \pm \sqrt{144 - 36}}{2} = 6 \pm 3\sqrt{3}.$$

12. **($4\sqrt{3}$)** Since $\triangle ABC \sim \triangle ADB$, we have $AB/AD = AC/AB$, or $AB^2 = (AD)(AC)$. Thus, $AB = \sqrt{(6)(8)} = 4\sqrt{3}$.