2002 National Mu Alpha Theta Convention
Mu Division–Number Theory Topic Test Solutions

1. **A** The primes between 100 and 120 are 101, 103, 107, 109, 113.

2. **A** Since 5! divides all three and and no number larger does (since no larger number divides 5!), 5! is the greatest common factor.

3. **B** Since $121 = 11^2$, 121 has 3 divisors (1, 11, 121).

4. **D** Rearranging, we have $n^2 - m^2 = 49$, so $(n-m)(n+m) = 49$. Thus, either $n-m = n+m = 7$, $n+m = 49$ and $n-m = 1$, or $n+m = 1$ and $n-m = 49$. The first and the last cases yield nonpositive solutions for either n or m, so they must be excluded. The middle case gives $(m, n) = (24, 25)$.

5. **E** Letting $i = a^2$, $j = b^3$, and $k = c^4$, it's apparent that the product $a^2b^3c^4$ need not be any integer power of any integer.

6. **A** 2^a can only divide 3^b if it has no factors besides 1 and powers of 3; therefore, $a = 0$.

7. **B** Since $89^2 < 20^3 < 90^2$ and $96^2 < 21^3 < 97^2$, there are $96 - 90 + 1 = 7$ perfect squares between 20^3 and 21^3.

8. **C** Since m^2 is odd, m is odd. Let $m = 2k + 1$. Thus,

$$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4n + 3.$$

Rearranging the final equality and dividing by 2 gives us

$$2(k^2 + k - n) = 1,$$

which has no solutions in integers.

9. **B** 999 and 998 are not prime. 997 is.

10. **A** The GCM of 2^8, $(2^3)(3)$ and $(2^2)(11)$ is $(2^8)(3)(11) = 8448$.

11. **D** $7 \cdot 1 = 7$, remainder 2 when divided by 5; $7 \cdot 2 = 14$, remainder 4; $7 \cdot 3 = 21$, remainder 1; $7 \cdot 4 = 28$, remainder 3.

12. **A** Numbers with 9 divisors must be of the form p^8 or p^2q^2, where p and q are primes. Since $2^8 > 200$, there are no answers of this form. For the other, we have $2^2 \cdot 3^2 = 36$, $2^2 \cdot 5^2 = 100$, $2^2 \cdot 7^2 = 196$. The sum of these is 332.

13. **D** Since $9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$, it has $8 \cdot 5 \cdot 2 \cdot 2 = 160$ divisors.

14. **B** Consider the table below, with values of n across the top and m along the side and $5n + 7m$ in the middle:

	1	2	3	4	5	6	7
1	12	17	22	27	32	37	42
2	19	24	29	34	39	44	49
3	26	31	36	41	46	51	56
4	33	38	43	48	53	58	63
5	40	45	50	55	60	65	70

Thus, we can form any number k such that $k \equiv 2 \pmod 5$ and $k \geq 12$, any $k \equiv 4 \pmod 5$ and $k \geq 19$, any $k \equiv 1 \pmod 5$ and $k \geq 26$, $k \equiv 3 \pmod 5$ and $k \geq 33$, any $k \equiv 0 \pmod 5$ and $k \geq 40$. Any number in these congruence classes below the stated minimum cannot be formed. Thus, 35 is the largest that cannot be formed. (Note that the above sketches out a general proof that the largest number that cannot be written in the form $km + jn$ for positive m, n is kj if $(k, j) = 1$, and that the largest with nonnegative m, n is $kj - k - j$.)

15. **C** Simplifying each gives us:

$$x = 2^{2^{2^{2^2}}} = 2^{2^{16}}, y = 3^{3^{3^3}} = 3^{3^{27}}, z = 4^{4^4} = 4^{2^8} = (2^2)^{2^8} = 2^{2^9}.$$

Thus, obviously $y > x$ since $27 > 16$ and $3 > 2$, and $x > z$ because $2^{16} > 2^9$. Thus, $z < x < y$.

16. **E** Since $20 = 2^2 \cdot 5$, we must find how many factors of 5 and of 2 are in 2002!. There's a factor of 5 for every multiple of 5 in the product 2002!, another for every multiple of 25, another for every multiple of 125, and so on, for a total of:

$$\lfloor \frac{2002}{5} \rfloor + \lfloor \frac{2002}{25} \rfloor + \lfloor \frac{2002}{125} \rfloor + \lfloor \frac{2002}{625} \rfloor = 499.$$

By the same method, we find that there are 1995 factors of 2 in 2002!, so 2^2 divides it $\lfloor \frac{1995}{2} \rfloor =$ 997 times. Thus, 5 is the limiting factor, so $n = 499$ is the largest permissible n.

17. **C** Consider the equation mod 9. Since $63n^3 \equiv 0 \pmod 9$ and $777700006 \equiv 7 \pmod 9$, we must find an m such that $m^3 \equiv 7 \pmod 9$. However, if we write $m = 3k + j$, where $j = 0, 1$, or 2, then

$$m^3 = (3k + j)^3 = 27k^3 + 27k^2 j + 9kj^2 + j^3.$$

The first three terms are divisible by 9, so $m^3 \equiv j^3 \pmod 9$. Since $j = 0, 1$, or 2, $m^3 \equiv 0, 1$, or 8 (mod 9) for all m. Thus, there are no solutions to the equation.

18. **A** Consider two integers $0 \leq k < j < 103$. We aim to prove that $j^2 \equiv k^2 \pmod{103}$ iff $j + k = 103$.

If $j^2 \equiv k^2 \pmod{103}$, then $j^2 - k^2 = 103m$ for some integer m. Thus, $(j - k)(j + k) = 103m$. Hence, either $(j - k)$ or $(j + k)$ is divisible by 103. Since $j \neq k$ and $0 \leq k < j < 103$, $0 < j - k < 103$ and $0 < j + k < 2 * 103$, so $j + k$ must equal 103. Similarly, we can show that any j, k such that $j + k = 103$ satisfies $j^2 \equiv k^2 \pmod{103}$. Hence, no two of the squares 0^2, 1^2, 2^2, ..., 51^2 are congruent mod 103, and all the squares from 52^2 through 103^2 are congruent to a sqaure in the first list. Any squares of larger integers are clearly congruent to one of the squares from 0^2 to 103^2, so they need not be considered. Thus, there are 52 different quadratic residues of 103.

19. **C** In algebraic terms, we have $6 * 10^n + k = 25 * k$, or $5^n \cdot 2^{n-2} = k$. Hence, our solutions are (for $n = 2, 3, 4, 5$): 625,6250,62500,625000.

20. **B** We seek integers n such that $1000 < n^2 < 10000$ and all digits of n^2 are even. First, n cannot be odd. Suppose n ends in 0. Quick examination yields $n = 80$ as the only solution of this form. Suppose $n = 10a + 4$. Thus, $n^2 = 100a^2 + 40a + 16$, hence the tens digit will always be odd. Similarly, we can dismiss n of the form $10a + 6$. This leaves us n that end in 2 or 8. Inspection reveals solutions $n = 92, 68, 78$, for a total of 4 solutions.

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21. **A** Rearranging yields $10(m+n) = mn$, or

$$(m-10)(n-10) = 100.$$

Since 100 has 9 divisors, there are 9 values $m-10$ can take on (with $n-10$ equal to $100/(m-10)$). Moreover, $m-10$ and $n-10$ could both be negative, yielding another 9 solutions. However, we must omit the solution $m = n = 0$ since that would give us $1/0 + 1/0 = 1/10$ as our initial problem. Thus, there are 17 solutions.

22. **C** Since $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, and $7^4 = 2401$, the last two digits of successive powers of 7 go in a cycle: $07, 49, 43, 01$. Thus, the problem is reduced to evaluating $7^7 \pmod 4$. Since $7 \equiv 3 \pmod 4$, the powers of 7 alternate $3, 1, 3, 1, \ldots \pmod 4$. Hence, $7^7 \equiv 3 \pmod 4$, so $7^{7^7} \equiv 43 \pmod{100}$ and the tens digit is therefore 4.

23. **B** Let there be k 9th graders, so there are $10k$ 10th graders and $n = 11k$ total students. The 9th graders play each other in $\binom{k}{2} = k(k-1)/2$ ways; no matter how these games turn out, they will collectively earn $k(k-1)/2$ points. Similarly, the 10th graders get $\binom{10k}{2} = 5k(10k-1)$ points from their games amongst themselves.

Now suppose that the 10th graders get a total of j points from the $10k^2$ games between 9th and 10th graders, leaving the 9th graders $10k^2 - j$ points from those games. Since the 10th graders have a total of 4.5 times as many points as the 9th graders, we have:

$$5k(10k-1) + j = 4.5(k(k-1)/2 + 10k^2 - j)$$
$$\therefore 200k^2 - 20k + 4j = 9k^2 - 9k + 180k^2 - 18j$$
$$\therefore 2j = k - k^2.$$

This final equation only has one solution for $j \geq 0, k > 0$, namely $(j, k) = (0, 1)$. Thus, the only possible value of n is 11.

24. **B** Clearly no m which are multiples of 3 can have powers which are only 1 more than a multiple of 3^{10}. For other m, we use Euler's generalization of Fermat's Theorem since $(m, 3^{10}) = 1$:

$$m^{\phi(3^{10})} \equiv 1 \pmod{3^{10}}$$

Since $\phi(3^{10}) = 3^{10}(1 - 1/3) = 3^{10} - 3^9$, we have

$$m^{3^{10}-3^9} \equiv 1 \pmod{3^{10}}$$

for all $m < 1000$ which are not divisible by three. There are $2/3(999) = 666$ such integers.

25. **D** Let $S(n)$ be the sum of the digits of n. Thus, we seek $S(S(S(4444^{4444})))$. Since $4444^{4444} < 10000^{5000}$, and 10000^{5000} is 1 followed by $4 \cdot 5000 = 20000$ zeroes,

$$S(4444^{4444}) < \underbrace{S(999\cdots999)}_{20000\ 9s} = 9 * 20000 = 180000,$$

. Since $S(4444^{4444}) < 180000$, $S(S(4444^{4444})) < S(99999) = 45$ because 99999 has the largest sum of digits of numbers less than 180000. Finally, $S(S(S(4444^{4444}))) < S(39) = 12$ because 39 has the largest sum of digits among numbers less than 45. Hence our answer is less than 12. To find the answer, we observe that $S(n) \equiv n \pmod 9$ for all n, (prove this by noting that $10 \equiv 1 \pmod 9$). Hence,

$$S(S(S(4444^{4444}))) \equiv 4444^{4444} \pmod 9 \equiv 7^{4444}.$$

The powers of 7 cycle $7, 4, 1, 7, 4, 1 \ldots \pmod 9$ and $4444 \equiv 1 \pmod 3$, so $7^{4444} \equiv 7 \pmod 9$. Since 7 is the only positive number less than 12 which is congruent to 7 mod 9,

$$S(S(S(4444^{4444}))) = 7.$$