Alpha Equations and Inequalities
Mu Alpha Theta National Convention 2003
SOLUTIONS

1. (C) Amplitude = $\sqrt{A^2+B^2} = \sqrt{5^2+9^2} = \sqrt{106}$
2. (B) X intercept at f(0).
   Therefore f(0+k)=f(0)f(k) » f(k)=f(0)f(k) » f(0) = 1
3. (C) The sum of complex = Real + Imaginary roots = -b/a = -(-5)/1 = 5
4. (C) To maximize $\frac{4}{3+\cos\theta}$, we must make the denominator as small as possible. Given that $0 \le \theta \le 2\pi$, $\cos\theta$ = -1 at smallest value.

   D(max) = $\frac{4}{3+(-1)} = \frac{4}{2} = 2$

   E(min) = $\frac{82+9x^2+9x}{3(3+x)} = \frac{9(3+x)^2+1}{3(3+x)} = y$ z=x+3

   $\frac{9z^2+1}{3(z)} = y \Rightarrow 9z^2 - 3zy + 1 = 0 \Rightarrow z = \frac{3y \pm \sqrt{9y^2-36}}{18} = \frac{y \pm \sqrt{y^2-4}}{6}$

   Therefore, for z to be real, $y \le -2 \cup y \ge 2$

   Numerator is always positive, *E(min)*=2

   <D,E>•<E,D> = <2,2>•<2,2> = 4+4 = 8
5. (B) $x_{symmetry} = \frac{-b}{3a} = \frac{4}{3}$ $y_{symmetry} = (\frac{4}{3})^3 - 4(\frac{4}{3})^2 + 7(\frac{4}{3}) - 4 = \frac{16}{27}$

   Point of symmetry = $(\frac{4}{3}, \frac{16}{27})$
6. (A)

Units		Gross
288B + 576P + 144A = 85		2B + 4P + 1A = 85
576B + 288P + 288A = 80	»	4B + 2P + 2A = 80
288B + 288P + 720A = 75		+ 2B + 2P + 5A = 75
		8B + 8P + 8A = 240

   which reduces to 2B + 2P + 2A = **60**

7. (E) x > e, Arctan(ln(e)) = Arctan(1) = $\frac{\pi}{4}$

   This is the lower bound of the interval

   $\lim_{x\to\infty} \arctan(\ln(\infty)) = \lim_{x\to\infty} \arctan(\infty) = \frac{\pi}{2}$

Range = ($\frac{\pi}{4},\frac{\pi}{2}$)

8. (B) $\frac{1}{(n+2)(n+3)}=\frac{1}{n+2}-\frac{1}{n+3}$

As this summation goes to infinity, the terms are

$(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+(\frac{1}{5}-\frac{1}{6})...+(\frac{1}{n+2}-\frac{1}{n+3})$

This is a telescoping sequence, which evaluates to

$\lim_{b\to\infty}\frac{1}{3}-\frac{1}{(n+3)}=\frac{1}{3}$.

9. (C) Given sec(x) = A, over $0\le x\le 2\pi$, x has 2 possible values. $|\sec x|$ has 4 possible solutions since the absolute value allows for sec(x) = -A. A, of course, is in the range of sec(x). Therefore, for any A, $|\sec x|$ =A has 4 solutions over $0\le x\le 2\pi$.

10. (D) For $(x^2-11x-11)^{(x^2+8x+15)}=1$ to hold true:

$x^2-11x-11=1$	$x^2+8x+15=0$
x = -1 or 12	**x = -5 or -3**

11. (D) This can be represented by the diophantine equation 11x+8y=345.

By testing the first few values of x, you get (3,39) is a solution. To determine all other possible points, we must add 8 to the x coordinate and subtract 11 from the y coordinate.

X	3	11	18	27
Y	39	28	17	6
+	42	39	36	33

No other solutions since Y can't be negative.

42+39+36+33 = 150

12. (D) I is false since all points of intersection of a function and its inverse have the same sign on the x and y coordinate.
II is false since points of intersection are not limited to lattice points
III is true
The answer is only III

13. (B) Let $n=\frac{3p+35}{2p-7}$ and $k=(2p-7)$. This gives $3p+35=kn$.

Using the equation of k and kn we simplify to eliminate the p's from one side of the equation. This gives the equation $k(2n-3)=91$. This can be expressed as either 1*91 or 7*13.

Solving for 2p-7 for values of 1, 91, 7 and 13 we get p=4, 49, 7, and 10. 4+49+7+10 = **70.**

14. (C) Since the bee turns around immediately when touching the other train, it can be visualized as going in a straight line until the two trains collide.

$$t_{collision} = \frac{4000m}{85\frac{m}{s} + 115\frac{m}{s}} = 20s \qquad x_{bee} = rt = (456\frac{m}{s})(20s) = 9120m$$

15. (B) Trial and error. f(x) +g(x) + h(x)

$= 4x^3 - 10x + \cos(x) + x^3 - 10x + 4\sin(x) + 7x^5 - \cos(x)$

$= 7x^5 + 5x^3 - 20x + 4\sin(x)$

Each monomial of this function is and odd function, therefore f(x) +g(x) + h(x) is an odd function.

16. (B) Triangular Number Sequence: 1,3,6,10,15,21,28…

$T_1 = 1$ $\qquad -T_4 = -10$

$T_5 = 15$ $\qquad T_6 = 21$

Area of bounded area = (15-1)(21+10) = **434**

17. (E) *m+2 < -2m +13 < 4m - 6*

Separate into 2 inequalities

m+2<-2m+13 $\qquad$ *-2m+13<4m-6*

3m < 11 $\qquad$ *19 < 6m*

$m < \frac{11}{3}$ $\qquad \frac{19}{6} < m$

Therefore $\frac{19}{6} < m < \frac{11}{3}$

18. (D) Arithmatic Mean = $\frac{A+B}{2}$ Geometric Mean = $\sqrt{AB}$

Harmonic Mean = $\frac{A+B}{2AB}$ Root Mean Square = $\sqrt{\frac{A^2+B^2}{2}}$

Given that B >A >1, The harmonic mean will always be less than the arithmetic mean because of the addition AB in the denominator. The geometric mean will also be less than the arithmetic mean because the square-root function grows much slower than a linear function.

The last to decipher which is greatest is the Arithmatic Mean and Root Mean Square. Let us do so by assuming one is greater, then proving or disproving that fact.

$\frac{A+B}{2} < \sqrt{\frac{A^2+B^2}{2}}$

$\frac{A^2+B^2+2AB}{4} < \frac{A^2+B^2}{2}$ $\qquad 2AB < A^2+B^2$

This is always true, therefore Root Mean Square is always greater than the Arithmatic mean.

19. (D) Pick a point on one of the lines and use the point-line distance formula :

y=4(0)+9 (0,9)

$\frac{|(9)-4(0)-11|}{\sqrt{1^2+4^2}} = \frac{2}{\sqrt{17}} = \frac{2\sqrt{17}}{17}$

20. (B) Billie Joe = $1000(1+\frac{.06}{4})^{4\cdot 10} = 1814.02$

Mike = $990(1+\frac{.065}{2})^{2\cdot 10} = 1876.88$

Tre = $980e^{.061\cdot 10} = 1803.62$

1814.02+(.81)1876.88+1803.62= 5137.91

$x+y=7xy$

21. (C) $(x+y)^2 = 49(xy)^2 = x^2+y^2+2xy$ x+y=A

$49A^2-2A-(x^2+y^2) = 49A^2-2A-\frac{37}{36}$

$A = \frac{2 \pm \sqrt{2^2-4(-\frac{37}{36})(49)}}{2\cdot 49} = \frac{2 \pm \frac{43}{3}}{2\cdot 49} = \frac{1}{6} or \frac{-37}{294}$

A must equal $\frac{1}{6}$ because x and y are positive

$\frac{x+y}{xy} = 7 = \frac{x+y}{\frac{1}{6}}$ $\qquad (x+y)^3 = (\frac{7}{6})^3 = \frac{343}{216}$

$\frac{7}{6} = x+y$

22. (A) A = $2^5 = 32$ since every iteration just multiplies the original coefficient of 2 by 2, and this happens 4 times

B = $((((3 \cdot 2+3) \cdot 2+3) \cdot 2+3) \cdot 2+3) = 93$ since every iteration modifies the constant by multiplying by 2 and adding 3, and this happens 4 times. A + B = 32 + 93 = 125

$$(x+\frac{1}{x})^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = x^3 + \frac{1}{x^3} + 3(x+\frac{1}{x}) = 3^3 = 27$$

23. (C) $x^3 + \frac{1}{x^3} + 3(3) = 27$

$$x^3 + \frac{1}{x^3} + = 27 - 9 = 18$$

24. (C) Since both function hold the same period and are sine and cosine, the x shift change is merely $2n\pi \pm \frac{\pi}{2}$. At n=0, the first shift is $\pm \frac{\pi}{2}$.
The y shift is then augments by where the function hits the y axis, -2-5=-7
(This is negative because since sine must shift down to get to the cosine function) Therefore, C is the answer.

25. (B) The floor function only yield integers, therefore, we must find all pairs of integers such that $x^2 + y^2 = 400$
By trial an error, there are 12 such pairs:
$(\pm 12, \pm 16), (\pm 16, \pm 12), (\pm 20, 0) \, and \, (0, \pm 20)$

Yet, since we are using the floor function, each lattice point that is a solution to $x^2 + y^2 = 400$ has an area of 1. This is because there are a number of points which, when evaluated through the floor function, can yield the solution points. In essence, we want the are of all points which satisfy
$(\lfloor x \rfloor, \lfloor y \rfloor) = (\pm 12, \pm 16), (\pm 16, \pm 12), (\pm 20, 0) \, or \, (0, \pm 20)$
For example, take the point (20,0). The point (20.1, 0.1) or (20.919191919,0.99292918) evaluate to (20,0) when the floor function is used. There is a unit square of points which satisfy this equation, 12 unit squares for all 12 points. Therefore, the area that satisfies the equation is 12.

26. (E) y = x, y= -x, 3x-y=8
Therefore, the 3 points of intersection (bounding the area) are (0,0), (2,-2), (4,4).
Area of triangle =

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   0 0
  0 2-2 0        8-(-8) = 8
 -8 4 4 8          2
  0 0 0  0
 -8      8
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27. (A) $\log_a b = 4$ which means $a^4 = b$

$\log b + \log a = \log a^4 + \log a = \log a^5 = 5$

$a = 10$

$b = 10^4 = 1000$

b-a = 1000-100 =990 = $2 \cdot 3^2 \cdot 5 \cdot 11$

Positive integral factors of b-a = (1+1)(2+1)(1+1)(1+1)=24

28. (C)

$x = \frac{kz^3}{y^2}$

$200 = \frac{k2^3}{5^2}$

$\therefore k = 625$

$x = \frac{625z^3}{y^2}$

$35 = \frac{625 \cdot 7^3}{y^2}$

$y^2 = \frac{625 \cdot 343}{35} = 6125$

$\therefore y = 35\sqrt{5}$

29. (A)

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + ... + \frac{1}{\log_{99} 100!} + \frac{1}{\log_{100} 100!} = m$$

$$\frac{\log 2}{\log 100!} + \frac{\log 3}{\log 100!} + \frac{\log 4}{\log 100!} + ... + \frac{\log 99}{\log 100!} + \frac{\log 100}{\log 100!} = m$$

$$\frac{\log 100!}{\log 100!} = m = 1$$

30. (C) $12x^2 - 11x - 15 \geq 0.$

$(4x+3)(3x-5) \geq 0.$

Critical points at $\frac{-3}{4}$ *and* $\frac{5}{3}$

Testing the different intervals:

$x < \frac{-3}{4}$	f(-2) = (4(-2)+3)(3(-2)-5) = 55 > 0 TRUE
$\frac{-3}{4} < x < \frac{5}{3}$	f(0) = 3(-5) = -15 > 0 FALSE
$\frac{5}{3} < x$	f(2) = (4(2)+3)(3(2)-5) = 11 > 0 TRUE

Solution Interval is $(-\infty, -\frac{3}{4}]$ U $[\frac{5}{3}, \infty)$