Alpha Applications Solutions
FAMAT State 2004

C 1. Period = $\frac{2\pi}{B}$, where $y = A\sin(Bx + C) + D$

B 2. Rewrite as x = $\frac{1}{2}\sin(3y)$; same formula applies as #1

D 3. Given $ax^2 + bx + c = 0$, the product of the roots = $\frac{c}{a}$, and the sum of the roots = $\frac{-b}{a}$; therefore, Andy reveals that $c = 12$, and Westerdale reveals that $b = -8$. So Christine's solution is the correct solution to $x^2 - 8x + 12 = 0$: $\{2,6\}$.

C 4. From 1:00 to 2:00, the first train covers 150 miles, leaving the distance between trains to be 450 miles. The distance between the trains shrinks at 250 miles per hour (sum of speeds of trains), so the time after 2:00 is $\frac{450}{250} = 1.8$ hours after 2:00, or 3:48 PM

B 5. Mr. Bantz walks for 20 seconds, so the fly goes for 20 seconds as well. Distance is just rate times time, so the fly travels 200 feet.

A 6. The expression factors as a perfect square trinomial.

C 7. The gender of the last two children is independent of the first three. So the probability that two children are both girls is the product of the probabilities that each one is a girl: $\frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$.

C 8. The expression factors as the difference of two perfect squares.

A 9. The problem is equivalent to finding the number of distinct arrangements of the following characters: xxxxxxxxxxxxxxx|||. Each "x" represents a snack and each "|" represents a partition. For example, xxxxx|xx|xxxxx|xxx would be an arrangement where Daniella receives 5 snacks, Arun receives 2 snacks, Abe receives 5 snacks, and Paula receives 3 snacks. Since each lunchbox receives at least one snack, we can fix one "x" in each section, leaving 11 "x"s and 3 "|"s that can move. Therefore there are ${}_{11+3}C_3$, or 364 possible arrangements.

C 10. The total distance that the ball travels is given by:

$$50 + 2(50)\left(\frac{3}{5}\right) + 2(50)\left(\frac{3}{5}\right)^2 + 2(50)\left(\frac{3}{5}\right)^3 + ... + 2(50)\left(\frac{3}{5}\right)^n + ...,$$ which is equal to

$$50 + 2\left(\frac{50\left[\frac{3}{5}\right]}{1 - \frac{3}{5}}\right) = 50 + 2\left(75\right) = 200 \text{ feet.}$$

B 11. The sum of the slopes of the asymptotes of any non-degenerate hyperbola is 0 (unless one of the slopes is undefined, but in this case both slopes are).

B 12. Subtract one from both sides, and factor out an x: $x\left(ax^5 + bx^3 - cx^2 - dx + e\right) = 0$. This equation has one root of zero, and by Descartes's sign rule, a maximum of 2 positive roots and a maximum of 1 negative root, for a total of 4 roots maximum.

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B 13. Notice the pattern: $2^1 \equiv 2 \bmod 7$, $2^2 \equiv 4 \bmod 7$, $2^3 \equiv 1 \bmod 7$, $2^4 \equiv 2 \bmod 7$, $2^5 \equiv 4 \bmod 7$, etc. Continuing the pattern, $2^{678} \equiv 1 \bmod 7$.

C 14. Reflect Mauro's house over $x=6$ to the point $(12,16)$. The distance between Mauro and the reflection of his house is $\sqrt{12^2+16^2}=20$.

E 15. The point is the centroid of the triangle, so the distance is 0.

D 16. The base 10 log of $2004^{2004} = 2004(\log[2004]) \approx 6617.003$. So $2004^{2004} \approx 10^{6617.003} = 10^{6617}\left(10^{.003}\right) \approx 1.007 \bullet 10^{6617}$, which has 6618 digits.

D 17. $\cos\theta = \dfrac{\vec{a}\bullet\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\langle 1,2,0\rangle \bullet \langle 2,0,4\rangle}{|\langle 1,2,0\rangle||\langle 2,0,4\rangle|} = \dfrac{2}{\sqrt{5}\sqrt{20}} = .2$ $\quad \cos^{-1} .2 \approx 1.37$

C 18. $4r^2\sin^2\theta - 16r^2\cos^2\theta = 64$ $\quad 4y^2 - 16x^2 = 64$ $\quad \dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$

$a=4$, $b=2$, and for a hyperbola $c=\sqrt{a^2+b^2}=2\sqrt{5}$. $e=\dfrac{c}{a}=\dfrac{2\sqrt{5}}{4}=\dfrac{\sqrt{5}}{2}$

A 19. Let $m = 3141592654$. The problem is now:
$(m+10)(m-10)-(m+20)(m-20)=(m^2-100)-(m^2-400)=300$

E 20. A zero is created for every power of 5 that $n!$ is divisible by because there are many more powers of 2 than 5. Therefore the number of zeros at the end of $n!$ is $\sum_{i=1}^{\infty}\left\lfloor \dfrac{n}{5^i} \right\rfloor$. So for n=2000, $\sum_{i=1}^{\infty}\left\lfloor \dfrac{2000}{5^i} \right\rfloor = 400+80+16+3+0+0+\ldots = 499$. Repeating the process for n=1999 yields only 498 zeros, so 2000 is the smallest n such that $n!$ ends with at least 499.

B 21. $M \times M^{-1} = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C 22. $x=\sqrt{2070+x}$ $\quad x^2-x-2070=0$ $\quad (x-46)(x+45)=0$ $\quad x=46$

B 23. expected value $=30\left[-1\left(\dfrac{4}{5}\right)+4\left(\dfrac{1}{5}\right)\right]$

A 24. Only $x^2+y^2=1$ and $x^2+y^2=2$ qualify.

C 25. Either the exponent is 0, yielding $\{5,-2\}$; the base is 1, yielding $\{-1,-4\}$; or the base is -1 and the exponent is even, yielding $\{-2,-3\}$. Thus the solution set is $\{-4,-3,-2,-1,5\}$.

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D 26. See FAMAT rules.

D 27. Utilizing several row reduction operations, the matrix simplifies to: $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$ So the determinant = 2(1)(3)(6)(1) = 36

B 28. Bearing is measured in degrees clockwise from north. The final coordinates of the boat are $\left(5\sqrt{3},0\right)$, which is $090°$ clockwise from north.

C 29. Making use of properties of geometric series in the last two lines:

$$\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\ldots+\frac{n}{3^n}+\ldots$$

$$=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots+\frac{1}{3^n}+\ldots$$

$$+0+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots+\frac{1}{3^n}+\ldots$$

$$+0+0+\frac{1}{27}+\frac{1}{81}+\ldots+\frac{1}{3^n}+\ldots$$

$$\vdots$$

$$=\frac{1/3}{1-1/3}+\frac{1/9}{1-1/3}+\frac{1/27}{1-1/3}+\ldots+\frac{1/3^n}{1-1/3}+\ldots$$

$$=\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\ldots+\frac{1}{2\left(3^n\right)}+\ldots=\frac{1/2}{1-1/3}=\frac{3}{4}$$

E 30. The solution is $\{\pm 7\}$, see FAMAT rules.