DIFFERENTIAL EQUATIONS – CALCULUS
FAMAT State Convention 2004

For all questions, E. NOTA means none of the above answers is correct. Unless otherwise stated, assume all numbers are real.

1. Which of the following best describes the differential equation $\frac{d^2y}{dt^2}+t\frac{dy}{dt}+t^2y=\sin\left(t^2+4t\right)$

   A. First order, linear B. First order, nonlinear
   C. Second order, linear D. Second order, nonlinear E. NOTA

2. The general solution of $y\,dx+x\,dy=0$ is a family of

   A. Circles B. Ellipses C. Hyperbolas D. Parabolas E. NOTA

3. Which of the following differential equations is exact?

   A. $3x^2\,dx+6xy\,dy=0$ B. $e^x y^2\,dx+2e^x y\,dy=0$
   C. $3x\sin y\,dx+3x\cos y\,dy=0$ D. $\ln y\,dx+e^x\,dy=0$ E. NOTA

4. Rubidium-83 has a half-life of 86 days. How long will it take for 10 grams of Rubidium-83 to decompose into 1 gram? Round your answer to the nearest day.

   A. 155 B. 210 C. 283 D. 319 E. NOTA

5. Find the general solution of the differential equation $xy'-y=y^2\sin x$.

   A. $y=\frac{x}{\cos x+C}$ B. $y=\frac{1}{x\left(\cos x+C\right)}$ C. $y=\frac{\cos x}{x}+Cx^{-1}$ D. $y=x\cos x+C$ E. NOTA

6. If $y'=\ln\left(x^y\right)$ and $y(1)=e$, find $y(2)$.

   A. 4 B. $4e$ C. $4e^2$ D. $4e^3$ E. NOTA

7. Find the general solution of $\left(x^2+y^2\right)dx+xy\,dy=0$

   A. $y^2=\frac{C}{x^4}-\frac{1}{2}$ B. $y^2=\frac{C}{x^2}-\frac{x^2}{2}$ C. $y^2=Cx^2e^{-4x}-\frac{x^2}{2}$ D. $y^2=\frac{C}{x}-\frac{2x^2}{9}$ E. NOTA

8. A box of mass 3 kg is being pushed with force $f(t)=\frac{1}{t+1}$, where t is the time measured in seconds and f is measured in Newtons. Suppose that the force due to friction is proportional to the velocity v of the box. The solution to which of the following differential equations will give a function for the velocity of the box at time t? Assume that k is constant.

   A. $v'=\frac{1}{t+1}$ B. $v'=\frac{1}{t+1}+kt$ C. $3v'=\frac{1}{t+1}-kv$ D. $v''=\frac{1}{t+1}-kv$ E. NOTA

9. If $dy = y^2x^3\,dx$ and $y(0)=1$, find $y(2)$.

A. -3 B. $-\frac{1}{3}$ C. $\frac{1}{3}$ D. 3 E. NOTA

10. Which of the following is a solution of $\left(y^2e^x + xy^2e^x + 3x^2\right)dx + 2xye^x\,dy = 0$?

A. $y^2e^x + x^3 = 4$ B. $xy^2e^x = 4$ C. $y^2 = \left(\frac{7}{x} - x^2\right)e^{-x}$ D. $y^2 = \left(\frac{1}{x} - 3x\right)e^{-x}$ E. NOTA

11. Use Euler's method with a step size of $\Delta x = .1$ to approximate $y(1.2)$ for the differential equation $y' = x + y$ given that $y(1) = 0$.

A. .21 B. .22 C. .23 D. .24 E. NOTA

12. April boils a cup of water and pours it into her mug to make some tea. The water is initially 212 degrees and will cool to 150 degrees after two minutes. If April wants to drink the tea when it is 90 degrees, how long will she have to wait after the water has been boiled? Round your answer to the nearest second. Assume that room temperature is constant at 70 degrees.

A. 5 minutes 39 seconds B. 6 minutes 23 seconds
C. 6 minutes 50 seconds D. 7 minutes 11 seconds E. NOTA

13. Find the general solution to the differential equation $2ye^{x^2}y' + 2xy^2e^{x^2} = x + 3$.

A. $y^2 = \left(\frac{x^2}{2} + 3x + C\right)e^{-x^2}$ B. $y^2e^{x^2} = C$
C. $y^2 = Ce^{x^2}\left(x^2 + 3x\right)$ D. $y^2 = (x + C)e^{-x^2}$ E. NOTA

14. A ball is thrown vertically into the air and hits the ground 2.5 seconds later. What is the maximum height of the ball in feet? Assume that air resistance is negligible and the acceleration due to gravity is $32\ ft/\sec^2$.

A. 25 B. 50 C. 75 D. 100 E. NOTA

15. Which of the following is an integrating factor that would make the differential equation $x^3(2y-3)dx + x^2dy = 0$ exact?

A. e^{x^2} B. $2x - \frac{2}{x}$ C. $e^{\frac{x^4}{2}}$ D. $\frac{e^{x^2}}{x^2}$ E. NOTA

16. Kyla borrows $10,000 to help pay for a new car with an annual interest rate of 6%. Assume the interest is compounded continuously and the payments are made continuously. If Kyla wants to pay off the loan in 4 years, how much money will she have to pay per month? Round your answer to the nearest cent.

A. $220.83 B. $234.33 C. $263.02 D. $264.84 E. NOTA

17. How many of the following are true about the differential equation $\left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^4y}{dx^4}\right) = 1$?

I. The differential equation is linear.
II. The degree of the differential equation is 2.
III. The order of the differential equation is 4.
IV. A solution to the differential equation is $y = \cos x$.
V. If $y = a(x)$ and $y = b(x)$ are solutions, then $y = a(x) + b(x)$ is also a solution.

A. 1 B. 2 C. 3 D. 4 E. NOTA

18. If $\frac{dy}{dx} = \sin(2x) + x$ and $y(0) = 0$, find $y\left(\frac{\pi}{2}\right)$.

A. $\frac{\pi^2 - 4}{8}$ B. $\frac{\pi^2}{8}$ C. $\frac{\pi^2 + 4}{8}$ D. $\frac{\pi^2 + 8}{8}$ E. NOTA

19. If $M(x,y)dx + N(x,y)dy = 0$ is an exact differential equation, then what is the value of $[M_y(x,y)]^3 - [N_x(x,y)]^3$?

A. 0 B. 1 C. 3 D. Cannot be determined E. NOTA

20. Find the general solution to the differential equation $x^2y' - y = -1$.

A. $y = 1$ B. $y = C$ C. $y = e^{1/x} + C$ D. $y = e^{-1/x} + C$ E. NOTA

21. A function $f(x)$ exists such that for all real *a*, $f'(a) = 2f(a)$. Given $f(0) = 1$, find $f(1)$.

A. 2 B. $2e$ C. $2e^2$ D. $2e^4$ E. NOTA

22. Ten pounds of salt is initially dissolved in a tank that contains 50 gallons of water. Water containing $\frac{1}{2}$ pounds of salt per gallon is flowing into the tank at a constant rate of 3 gallons per minute. The mixture is well stirred and flows out of the tank at 3 gallons per minute. If $y(t)$ is the pounds of salt in the tank after *t* minutes have elapsed, then which of the following differential equations represents the rate of change of salt in the tank?

A. $y' = \frac{3}{2}t$ B. $y' = \frac{3}{2} - \frac{3}{50}y$ C. $y' = \frac{3}{2}t - 3y$ D. $y' = \frac{3}{2}t - \frac{3}{50}y$ E. NOTA

23. Lauren is running for the president of her senior class, which contains 423 students. Suppose Lauren starts campaigning two weeks before the election (at which point she is her only supporter). The number of supporters she gains is proportional to the product of the number of supporters she has and the number that she does not have, and she is continuously gaining supporters. Assume that all of her supporters vote for her and that every student votes. If Lauren has 25 supporters one week before the election, how many votes should she expect to get? Round your answer to the nearest person.

A. 186 B. 197 C. 264 D. 368 E. NOTA

24. A monopolistic widget-producing company has an inverse demand function $P(Q) = 15000 - 1550Q$ where P is the price in dollars and Q is the quantity in thousands. The company's marginal cost is given by $MC(Q) = 6000 - 1600Q$ and has a fixed cost of \$2000. What is the maximum profit the company can earn? Assume that the company's revenue is $R(Q) = Q \cdot P(Q)$.

A. \$23600 B. \$25000 C. \$28600 D. \$34200 E. NOTA

25. The Verhulst Logistic Growth Equation is $\frac{dP}{dt} = rP - \frac{r}{K}P^2$, where $P(t)$ is the population at time t, r is the birthrate of the population under optimal conditions, and K is the carrying capacity. The equation was created to help explain the growth of a population in an ecosystem in which the population is limited by factors such as competition for food, land, etc. How many of the following statements are true concerning the Verhulst Equation? Assume r and K are positive.

I. The population is growing fastest when the population is half the carrying capacity
II. For small values of P, the population can be approximated by an exponential model
III. If $P > K$ the population will decrease
IV. $\lim_{t\to\infty} P(t) = K$

A. 1 B. 2 C. 3 D. 4 E. NOTA

26. Find the general solution to the differential equation $2x\sin y\,dx + \left(x^2\cos y + 2y\right)dy = 0$.

A. $x^2\sin y = C$ B. $2\sin y - x^2\sin y = C$
C. $x^2\cos y + y^2 = C$ D. $x^2\sin y + y^2 = C$ E. NOTA

27. The particle is moving along the y-axis with acceleration at time t given by $\frac{d^2y}{dt^2} = 3t\sin t$. If the particle is at $y = 2$ when $t = 0$ and is stationary when $t = \frac{\pi}{2}$, which of the following gives the position of the particle at time t?

A. $y = 3t\sin t + 6\cos t - 3t + 8$ B. $y = 3t\cos t - 6\sin t + 3t + 8$
C. $y = -3t\sin t - 6\cos t - 3t + 8$ D. $y = -3t\cos t + 6\sin t + 3t + 8$ E. NOTA

28. If $\zeta(x,y)=C$ is a solution to $f(x,y)\,dx+g(x,y)\,dy=0$, then what is a solution to $g(x,y)\,dx+f(x,y)\,dy=0$?

A. $\zeta(x,y)=C$ B. $\dfrac{1}{\zeta(x,y)}=C$ C. $\left[\zeta(x,y)\right]^2=C$ D. Cannot be determined E. NOTA

29. Find the orthogonal trajectories of the family of curves given by $y=\dfrac{C}{x^2}$.

A. $y=\dfrac{x^4}{8}+K$ B. $y=-\dfrac{1}{x}+K$ C. $y^2=\dfrac{x^2}{2}+K$ D. $y^2=-\dfrac{2K}{x^3}$ E. NOTA

30. Let $f(x)$ and $g(x)$ be twice differentiable functions such that $f(x)=xg(x)$ and $g'(x)=f(x)$. If $f'(x)=h(x)\cdot g(x)$, then what is $h(x)$?

A. $x+1$ B. x^2+1 C. x^2+x D. x^3+x E. NOTA