Sequences and Series—SOLUTIONS *State Convention 2004*

1. C—If $|r|>1$*, then it diverges (b/c its unbounded) and if* $|r|<1$ *then the limit = 0, hence converges. If* $r=1$*, converges to 0 and if* $r=-1$*, it diverges by oscillation. So* $-1<r\le 1$.

2. B--$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n(n+1)}$. *So,* $S_n=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-...+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}$. *And,* $\lim_{n\to\infty}S_n=1$.

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3. D—The limit of the ratio for the series $\sum\frac{1}{n^{3/2}}$ *is 1,so this test fails.*

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4. D--$S=\frac{a}{1-r}=\frac{2/3}{1-2/3}=2.$

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5. C--$\lim_{n\to\infty}(n+1)(x-3)=\infty$ *(diverges), unless x = 3.*

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6. D—The series is $\cos\frac{\pi}{4}-(x-\frac{\pi}{4})\sin\frac{\pi}{4}-\frac{(x-\frac{\pi}{4})^2}{2!}\cos\frac{\pi}{4}+\frac{(x-\frac{\pi}{4})^3}{3!}\sin\frac{\pi}{4}-...$ *And the coefficient of the required term is* $\frac{\sqrt{2}}{12}$.

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7. A—Using the series for e^x *and letting x = -0.1 gives the answer.*

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8. A--$f(x)=x\ln x;\ f^{/}(x)=1+\ln x;\ f^{//}(x)=\frac{1}{x};\ f^{///}(x)=\frac{-1}{x^2};\ f^4(x)=\frac{2}{x^3};\ f^5(x)=\frac{-3\cdot 2}{x^4}$ and $f^5(1)=-3\cdot 2.$.

So, the coefficient of $(x-1)^5=\frac{-3\cdot 2}{5!}=\frac{-1}{20}$.

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9. C—Using the Ratio Test, we get $\lim_{n\to\infty}\left|\frac{x}{2}\left(1+\frac{1}{n}\right)^n\right|=\left|\frac{x}{2}\cdot e\right|$. *So, the series converges when* $\left|\frac{x}{2}\cdot e\right|$ *< 1, that is, when* $|x|<\frac{2}{e}$. *So, the radius of convergence is* $\frac{2}{e}$.

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10. D—Since the ratio r < 1, the sum of the series = $\frac{a}{1-r}$ *or* $\frac{\pi^3}{3^\pi}\cdot\frac{1}{1-\frac{\pi^3}{3^\pi}}$*, which simplifies to get the answer.*

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11. A—The error of this given series is less in absolute value than the first term dropped, that is, less than $\left|\frac{(-1)^{300}}{900-1}\right|\approx 0.0011$. *The closest answer to this is A.*

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12. E—Using the Ratio Test, we get $\lim_{n\to\infty}\frac{1}{n+1}|x-1|$. *This equals 0 if* $x\ne 1\Rightarrow$ *convergent everywhere. It also converges if x = 1.*

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13. C—The power series for $\ln(1-x)$*, if x < 1, is* $-x-\frac{x^2}{2}-\frac{x^3}{3}-...$ *and the coefficient of the* x^3 *term is* $\frac{-1}{3}$.

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14. A—Rewrite the given limit as: $\lim_{n\to\infty}\left[\left(\frac{1}{n}\right)^{1/2}+\left(\frac{2}{n}\right)^{1/2}+\left(\frac{3}{n}\right)^{1/2}+...+\left(\frac{n}{n}\right)^{1/2}\right]\frac{1}{n}$. *Note that the interval in this case is 0 to 1 and* $\Delta x=\frac{1}{n}$, $x_k=\frac{k}{n}$*, and* $f(x_k)=(x_k)^{1/2}$. *This gives the integral.*

15. B—Differentiating, the new series is $1+\frac{x-1}{2}+\frac{(x-1)^2}{3}+\frac{(x-1)^2}{4}+\ldots$ Using Ratio Test gives the answer.

16. B—The limit approaches 0 as n gets large (denominator gets huge).

17. D—Using the p-Series test, where $p > 1$,as in D, will show convergence.

18. B—Using the Integral test for B, shows divergence.

19. D—Using the Limit Comparison Test shows convergence for this series/

20. A—Since the given series is a convergent geometric series with a = 2 and r = 0.2, we have

$$R_3=\frac{ar^3}{1-r}=\frac{2(1/5)^3}{1-1/5}=0.02$$

21. D—We want $\frac{2(1/5)^n}{1-1/5}<0.0002$. Solving, we get $n > 5.86$. so 6 terms are required for stated accuracy.

22. D—Rationalizing the denominators, we have $(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\ldots+(\sqrt{25}-\sqrt{24})=4$.

23. C—Subtract 2 from the beginning index of summation means add 2 to the formula that is summed.

24. B—We have $(t_{10}+t_2+\ldots+t_{n-1})+t_n=n^2t_n \Rightarrow (n-1)^2t_{n-1}=(n^2-1)t_n \Rightarrow t_n=\frac{n-1}{n+1}t_{n-1}$. So the sum is

$$50^2\bullet\frac{1}{50\bullet 51}=\frac{50}{51}.$$

25. B—Write the series as $1+3x+5x^2+7x^3+\ldots$ which then can be written as

$$[1+x+x^2+x^3+\ldots]+[2(x+x^2+x^3+\ldots)]+[2(x^2+x^3+\ldots)]+\ldots=\left[\frac{1}{1-x}\right]+\left[\frac{2x}{1-x}\right]+\left[\frac{2x^2}{1-x}\right]+\ldots=\frac{1+x}{(1-x)^2}$$

Letting $x=\frac{1}{3}$ gives the answer 3.

26. B—Plugging in, one can readily see that everything cancels but $2(n+1)-1=2n+1$.

27. B—Let $S=\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}+\ldots$ and $\frac{1}{5}S=\frac{1}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\ldots$ Subtracting these quantities gives

$\frac{4}{5}S=\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots$ or (using the infinite geom. sum formula), $\frac{4}{5}S=\frac{1/5}{1-1/5}\Rightarrow S=\frac{5}{16}$.

28. C—Rewrite the series as $1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}+\ldots=1-\frac{1}{201}=\frac{200}{201}$.

29. D—Using the geometric sum formula, we get $S=(1+i)^{22}-1$ and $(1+i)^2=2i$, so

$$(1+i)^{22}=(2i)^{11}\Rightarrow S=-1-i\bullet 2^{11}=-1-2048i$$

30. D—Using the formula given, when n = 1, $x_2=3$ and when n = 2, $x_3=5$.