2004 National Mu Alpha Theta Convention
Mu Division–Number Theory Topic Test

1. **B** The GCD is 6, so the LCM is $72 \times 30/6 = 360$.

2. **B** The smallest is the product of the three smallest primes: $(2)(3)(5) = 30$.

3. **A** If 0 were positive, then $(-4) \times 0 = 0$ provides a contradiction of I. The others remain true.

4. **D** Since $6 + A + 4 = 10 + A$, we find that the desired values of A are 2, 5, and 8. These have sum 15.

5. **B** 97 is the largest prime in the product $100 \times 99 \times 98 \times \cdots \times 1$.

6. **B** The sum of the first n positive integers is $n(n+1)/2$. The smallest n for which this is divisible by 13 is $n = 12$.

7. **D** There are several possible answers, such as $k + j = 3 + 1 = 4$ or $k + j = 3 + 4 = 7$, and so on.

8. **B** The number $n! + 1$ is divisible by n for $n = 1$. For all other n, $n! + 1$ leaves a remainder of 1 when divided by n.

9. **A** The highest is $1111111_2 = 127$, the lowest is $1000000_2 = 64$, for a total of 64 numbers.

10. **B** Numbers without circles are of the form pq where p and q are primes (not necessarily distinct) greater than 7. There are 16 such numbers less than 400.

11. **A** All but 0 and 4 are easily dismissed by noting that only 00 or 44 could be repeated last 2 digits. An ending of 4444 can be dismissed by noting that any such number is of the form 16k+12, which cannot be a perfect square.

12. **A** Either p or q must be even. Since $p < q$ and 2 is the only even prime (and the smallest prime), then $p = 2$.

13. **D** $AB + BA = AA + BB = 11(A + B)$, so all such sums must be divisible by 11.

14. **A** There are $120(1 - 1/2)(1 - 1/3)(1 - 1/5) = 32$ such numbers.

15. **C** Since $n^5 - 5n^3 + 4n = (n-2)(n-1)(n)(n+1)(n+2)$, we know the product is divisible by 3, 5, and 8. (Note that for $n = 3$, our product equals 120.)

2004 National Mu Alpha Theta Convention
Mu Division–Number Theory Topic Test

16. **B** There are 60 numbers from 10 to 99 which are divisible by 2 or by 3 (or both), so the probability is $60/90 = 2/3$.

17. **D** We have $(x+y)(x-y) = 105$. Since $105 = 105 \times 1 = 35 \times 3 = 21 \times 5 = 15 \times 7$, there are 4 sets of solutions. Each set gives 4 solutions (since we can change the sign of either x or y in a solution to give another solution), so there are 16 solutions.

18. **B** The answer is the least common multiple of the three numbers, which is 360.

19. **B** Only squares of composites are divisible by at list 4 distinct positive numbers. There are 13 of these less than 500.

20. **D** $420 = 42 * 10$, $294 = 42 * 7$. Therefore, $m + n = 420a + 294b = 42(10a + 7b)$.

21. **C** Multiply the three to get $a^2b^2c^2$ is divisible by $2^6 \times 3 \times 5 \times 7^2$. Therefore, abc must be divisible by 2^3, 3, 5, and 7, so it must be divisible by 840.

22. **B** $100A + 10B + C$ is either 900 or 360, so $A + B + C = 9$.

23. **A** There's a one to one correspondence between base 3 numbers without 2's and base 2 numbers. Therefore, we interpret every base 3 number without 2's from 1_3 to $1000000_3 = 729$ as base 2 number and get $1_2 = 1$ through $1000000_2 = 64$. We omit the last, since we want the numbers less than 729, for a total of 63.

24. **A** $720 = 3^2 \times 2^4 \times 5$ and $180 = 3^2 \times 2^2 \times 5$, so k must have 2^4 as a factor. It can have anywhere from 0 to 2 factors of 3 and 0 or 1 factors of 5, so there are 6 possibilities.

25. **B** Rearrange to find $y = 10x/(x-10)$, from which it follows that our maximal sum occurs when $(x, y) = (11, 110)$ or $(110, 11)$.

26. **D** The statement is true for all k. Consider the powers of 2 $\pmod{k}$. By the Pigeonhole Principle, at least two are the same, so we have $2^a - 2^b \equiv 0 \pmod{k}$ for those two.

27. **A** The left side is divisible by 3 and the right isn't. Therefore, there are no integer solutions.

28. **C** We consider each of the cases $n = 2, 3, \ldots, 9$ and find that there are 24 such pairs.

29. **B** Note that n is among the set of cycles of n. We will show that every 10-digit multiple of n that is a multiple of 11111 is such that all the members of the set of cycles of n are multiples of 11111. Let

$$n = a \cdot 10^9 + b,$$

where b is a 9-digit integer. Then, the last 'cycle' member is given by

$$10b + a = 10n - a \cdot 10^{10} + a = 10n - a(10^{10} - 1).$$

Since $10^{10} - 1 = 9999999999$, $11111|9999999999$, and $11111|n$, we know $11111|a(10^{10} - 1)$ and $11111|10n$, so $11111|10b+a$. Thus, the members of the set of cycles of n will all be divisible by 11111 if n is, and will not otherwise. Since the 10-digit multiples of n range from $90000 * 11111 + 11111 = 100001111$ to $900009 * 11111 = 9999999999$, there are $900009 - 90001 + 1 = 810009$ values of n which satisfy the problem.

30. **B** Since $f(n)$ equals the number of 1's in the binary representation of n, there are 5 numbers less than 2003 with $f(n) = 10$ (Consider a binary number with 11 digits, one of which is 0. There are 11 such numbers; 6 of these are greater than 2002.)