2004 National Mu Alpha Theta Convention
Mu Alpha Theta Olympiad

Provide full solutions to each of the problems below. Partial credit will be awarded. No credit will be given to answers that are given without justification.

1. Let K be a 7-member subset of the smallest 36 positive integers. Show that for every K we can find 4 members of K such that the product of two of them minus the product of the other two is divisible by 5.

2. Let P be a point randomly chosen inside tetrahedron $ABCD$. Let a_1, a_2, a_3, a_4 be the distances from P to the faces of $ABCD$. Let b_1, b_2, b_3, b_4 be the lengths corresponding altitudes of the tetrahedron (i.e. if a_i is the distance from P to a face of $ABCD$, then b_i is the length of the altitude to that face). Prove that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} = 1$$

3. Given $z = 3 + 4i$, find with proof the imaginary number w such that $|w| = 10$ and $|w - z|$ is maximal. (The magnitude of a complex number is determined as follows: if $z = a + bi$, where a and b are real, then $|z| = \sqrt{a^2 + b^2}$.)

4. Point D is on side BC of triangle ABC such AD bisects $\angle CAB$. Line l pases through A and is tangent to the circle through the vertices of $\triangle ABC$. Prove that the straight line through D parallel to l is tangent to the circle that is inscribed in triangle ABC.

5. Consider the polynomial $f(x)$ with degree n and integer coefficients. Given that

$$f(1) = f(2) = f(3) = f(4) = 2004,$$

prove that there is no integer m such that $f(m) = 1$.