2004 National Mu Alpha Theta Convention
Mu Alpha Theta Olympiad

1. We proceed by contradiction, showing that it is impossible to construct a set K such that there are not 4 members of K such that the product of two of them minus the product of the other two is divisible by 5. First, we cannot have 2 members of K which are divisible by 5. Hence, we assume that there is at most 1 member of K which is divisible by 5. Next, consider the members of K modulo 5. At most 1 mod can exist more than once in K, since if $w, x, y, z \in K$, $w \equiv z \pmod 5$, and $x \equiv y \pmod 5$, then $wx - yz \equiv 0 \pmod 5$. Furthermore, we can't have 4 members of K which are equivalent mod 5. Hence, we must only show that there cannot be 3 members of K which are equivalent mod 5. Finally, since $(4)(3) - (1)(2) \equiv 0 \pmod 5$, we can't have one of each nonzero class. Since only one number can be repeated, and it can't be repeated more than twice, and we can't have at least 1 of all 5 equivalence classes, we can have at most 6 members of K (the three repeats plus one from each of the other equivalence classes).

2. Let $[[ABCD]]$ be the volume of $ABCD$. We have

$$\frac{[PABC]}{[ABCD]} = \frac{a_1}{b_1}; \frac{[PABD]}{[ABCD]} = \frac{a_2}{b_2}; \frac{[PACD]}{[ABCD]} = \frac{a_3}{b_1}; \frac{[PBCD]}{[ABCD]} = \frac{a_4}{b_1}$$

Summing these four equations gives the desired

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} = 1.$$

3. Let $w = x + yi$. Since $|w| = 10$, we know that (x, y) is on the circle $x^2 + y^2 = 100$. To maximize $|w - z|$, we must select the point on this circle that is farthest from $z = (3, 4)$. To find this point, we construct the line through z and the origin. Where this line meets the circle on the opposite side of the origin from z is our desired point. Our line has equation $y = 4x/3$. Substituting this into our circle equation yields

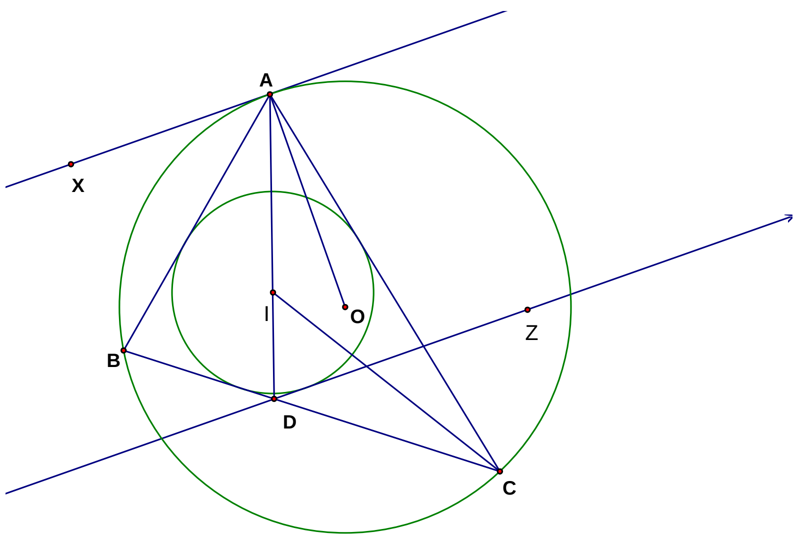
$$x^2 + \frac{16x^2}{9} = \frac{25x^2}{9} = 100.$$

Solving, we have $x = \pm 6$. Given that we want the point on the far side of the origin from z, we have as our solution $w = -6 - 8i$.

4. In the diagram below, let point X be on line l and Z be on the line through D parallel to l. We have

$$\begin{aligned}\angle ADZ &= \angle DAX \\ &= \angle BAX + \angle DAB \\ &= \angle BAX + \frac{A}{2} \\ &= \frac{\widehat{AB}}{2} + \frac{A}{2} \\ &= \angle ACB + \frac{A}{2} \\ &= \angle ADB\end{aligned}$$

Since line DB is tangent to circle I by definition, and $\angle IDB = \angle IDZ$, line DZ is also tangent to the circle.



5. Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

Thus,

$$\begin{aligned}f(m) - f(k) &= a_n(m^n - k^n) + a_{n-1}(m^{n-1} - k^{n-1}) + \cdots (m - k) \\ &= (m - k)g(m, k),\end{aligned}$$

where $g(m, k)$ is an integer. From the given information, we have $2003 = (m - k)g(m, k)$ for $k = 1, 2, 3, 4$ for some m. Since there are not 4 consecutive integers which divide 2003, these equations cannot all hold. Hence, we cannot have $f(m) = 1$ for any integer m.