Alpha School Bowl Solutions

1. Area of a rhombus with diagonals d_1 and d_2 is given by $\frac{1}{2}d_1 \cdot d_2$.

   $\frac{1}{2} \cdot 5 \cdot 12 = \boxed{30}$

2. To find the minimum of a parabola, complete the square:

   $f(x) = 2x^2 + 8x = 2[x^2 + 4x] = 2[(x+2)^2 - 4] = 2(x+2)^2 - 8$

   The minimum value is attained when $x = -2$ and $f(x) = \boxed{-8}$.

3. To find the area of a triangle given three side lengths, use Heron's formula. $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.

   $s = \frac{5+9+10}{2} = 12$ so $A = \sqrt{12(7)(3)(2)} = \boxed{6\sqrt{14}}$.

4. $P = \pi r + 2r$ for semicircle.

   $P = 6 + 3\pi \approx 6 + 3 \cdot \frac{22}{7} = \boxed{\dfrac{108}{7}}$.

5. Sum of infinite geometric with first term a and ratio r is $\frac{a}{1-r}$.

$$\begin{aligned} \frac{a}{1-r} &= 4a \\ \frac{1}{1-r} &= 4 \\ 1-r &= \frac{1}{4} \\ r &= \boxed{\frac{3}{4}} \end{aligned}$$

6. Magnitude of $\langle a, b\rangle = \sqrt{a^2+b^2}$.

   $|\langle\sqrt{12}, 5>\rangle| = \sqrt{\sqrt{12}^2 + 5^2} = \boxed{\sqrt{37}}$

7. The modem goes offline 10 minutes every 60 minutes and thus has an effective rate of $\frac{5}{6}$ its actual rate.

   $\frac{5}{6} \cdot 48 = \boxed{40}$

8. $g(3) = 3^{-2} + 3^{-3} + 3^{-5} = \frac{1}{9} + \frac{1}{27} + \frac{1}{243} = \frac{27+9+1}{243} = \boxed{\dfrac{37}{243}}$

9. The magnitude of the quotient of two complex numbers is equal to the quotient of the individual magnitudes. Since both numbers have the same magnitude, the answer is $\boxed{1}$.

10. There are 20^2 1x1 squares, 19^2 2x2 squares, $\ldots$, 2^2 19x19 squares and 1^2 20x20 square.

    $\sum_{n=1}^{20} n^2 = \frac{20 \cdot 21 \cdot 41}{6} = \boxed{2870}$

11. The matrix is in upper triangular form, so its determinant is the product of the elements along the main diagonal.

$$\begin{aligned} x(x-1) \cdot 1 \cdot 5 &= 100 \\ x^2 - x &= 20 \\ (x-5)(x+4) &= 0 \\ x &= \boxed{5, -4} \end{aligned}$$

12. The sum of the positive integral factors of a number with prime factorization $p_1^{k_1} \cdot p_2^{k_2} \cdots p_n^{k_n}$ is given by $\left(\frac{p_1^{k_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{k_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_n^{k_n+1}-1}{p_n-1}\right)$.

$210 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$, so the sum of factors is $\left(\frac{4-1}{2-1}\right)\left(\frac{9-1}{3-1}\right)\left(\frac{25-1}{5-1}\right)\left(\frac{49-1}{7-1}\right) = 3 \cdot 4 \cdot 6 \cdot 8 = \boxed{576}$.

13. The area of a regular hexagon with side length s is the same as the area of six equilateral triangles with side length s. The hexagon then has area $6 \cdot \frac{4^2\sqrt{3}}{4} = 24\sqrt{3}$. The triangle has area $\frac{2^2\sqrt{3}}{4} = \sqrt{3}$. The ratio of the two is $24:1$ or simply $\boxed{24}$.

14. With only five terms, it is easiest to add the values directly. $1+3+6+10+15 = \boxed{35}$.

15.

$$\begin{aligned}
54 - 54e^{\frac{-t}{4}} &= 34 \\
e^{\frac{-t}{4}} &= \frac{10}{27} \\
e^{\frac{t}{4}} &= 2.7 \\
\frac{t}{4} &= \ln(2.7) \\
t &= 4\ln(2.7)
\end{aligned}$$

$\ln(2.7)$ is very nearly 1, so $t \approx \boxed{4}$.

16. $P(\text{heads}) = 2P(\text{tails})$ so $P(\text{heads}) = \frac{2}{3}$ and $P(\text{tails}) = \frac{1}{3}$.

$P(\text{exactly 3 heads in 5 flips}) = {}_5C_3 \cdot P(\text{heads})^3 \cdot P(\text{tails})^2 = \frac{10 \cdot 2^3}{3^5} = \boxed{\frac{80}{243}}$.

17. It takes 30 minutes for the leak to drain the full bathtub completely and it takes 10 minutes to fill the tub while working against the leak. Let F equal the time to fill the tub (in minutes) if there were no leak:

$$\begin{aligned}
\frac{1}{F} - \frac{1}{30} &= \frac{1}{10} \\
\frac{1}{F} &= \frac{2}{15} \\
F &= \boxed{\frac{15}{2}}
\end{aligned}$$

18. In base 10, $CD_{11} = 11C + D$ and $DC_9 = 9D + C$.

$$\begin{aligned}
11C + D &= 9D + C \\
10C &= 8D \\
5C &= 4D
\end{aligned}$$

$C = 4$ and $D = 5$, so $D - C = \boxed{1}$.

19.

$$\begin{aligned}
4+\frac{3}{2+\frac{1}{x}} &= x \\
4+\frac{3x}{2x+1} &= x \\
\frac{3x}{2x+1} &= x-4 \\
3x &= (2x+1)(x-4) \\
2x^2-10x-4 &= 0 \\
x^2-5x-2 &= 0 \\
x &= \frac{5\pm\sqrt{33}}{2}
\end{aligned}$$

We have to discard the negative solution because we have only positive terms in the continued fraction, so the only valid answer is $\boxed{\frac{5+\sqrt{33}}{2}}$.

20. A 40° sector is $\frac{1}{9}$ the area of the entire circle.

$$\begin{aligned}
\frac{1}{9}\pi r^2 &= 12\pi \\
r^2 &= 108 \\
r &= 6\sqrt{3}
\end{aligned}$$

$C = 2\pi r = \boxed{12\pi\sqrt{3}}$.

21. It takes $\frac{12}{2} = 6$ seconds for the swimmer to cross. In that time, he has traveled 12 feet north and $4\cdot 6 = 24$ feet west, placing him $\sqrt{12^2+24^2} = 12\sqrt{5}$ feet from his starting point.

22.

$$\begin{aligned}
\sin(2x)+2\sin(x) &= \cos(x)+1 \\
2\sin(x)\cos(x)+2\sin(x) &= \cos(x)+1 \\
2\sin(x)(\cos(x)+1) &= \cos(x)+1 \\
(2\sin(x)-1)(\cos(x)+1) &= 0
\end{aligned}$$

On the interval, $\sin(x) = \frac{1}{2}$ when $x = \frac{\pi}{6}, \frac{5\pi}{6}$. $\cos(x) = -1$ only when $x = \pi$. The complete solution set on the interval is $\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \pi}$.

23. $H(x) = 1 - \frac{1}{x}$. To find the inverse, we solve:

$$\begin{aligned}
x &= 1-\frac{1}{H^{-1}(x)} \\
1-x &= \frac{1}{H^{-1}(x)} \\
H^{-1}(x) &= \frac{1}{1-x}
\end{aligned}$$

To find where the value of the function equals the value of the inverse, we solve:

$$\begin{aligned} 1-\frac{1}{x} &= \frac{1}{1-x} \\ x-1 &= \frac{x}{1-x} \\ (1-x)(x-1) &= x \\ x^2-x+1 &= 0 \end{aligned}$$

This equation has no **real** solutions, so $\boxed{\text{no values of } x}$ satisfy the equation.

24. The fifth roots of $3i+4$ can be found by solving the equation $x^5-(3i+4)=0$. The product of the roots of an odd-degree polynomial is found by negating the constant term and dividing it by the leading coefficient. $-\frac{-3i-4}{1}=\boxed{3i+4}$.

25. By definition, $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin(t)$ and $|\vec{u}\cdot\vec{v}|=|\vec{u}||\vec{v}|\cos(t)$. If $|\vec{u}\times\vec{v}|=|\vec{u}\cdot\vec{v}|$ then $\cos(t)=\sin(t)$. This occurs when $t=\frac{\pi}{4},\frac{5\pi}{4}$, when $\cos(t)=\pm\sqrt{2}$. The product of these values is $\boxed{-\frac{1}{2}}$.

26. Eliminate x from equation 1 by adding -5 times equation 2. Eliminate x from equation 3 by adding -2 times equation 2. We are left with:

$$\begin{aligned} 9y-12z &= 4 \\ 7y-4z &= 2 \end{aligned}$$

Eliminate z from equation 2 by adding -3 times equation 3. We end up with $-12y=-2\Rightarrow y=\frac{1}{6}$. It follows that $z=\frac{\frac{7}{6}-2}{4}=-\frac{5}{24}$ and $x=-1+\frac{1}{6}+\frac{15}{24}=-\frac{5}{24}$. $(x,y,z)=\boxed{\left(-\frac{5}{24},\frac{1}{6},-\frac{5}{24}\right)}$.

27. $24!+25!+26!=24!+25\cdot 24!+26\cdot 25\cdot 24!=676\cdot 24!$. Trailing zeros (in base 10) occur when the factors two and five are present together. There are always more twos than fives in a given factorial, so it is sufficient to count fives. 676 does not bring any additional fives so we can just count the fives in 24!. There are $\lfloor\frac{24}{5}\rfloor=4$ fives, and thus $\boxed{4}$ trailing zeros.

28. The ratio of the surface areas of two similar solids is equal to the square of the ratio of the lengths of their corresponding sides. $\frac{1}{1200}^2=\boxed{\frac{1}{1440000}}$.

29. Rewrite the equations $x-2y=2$ and $(x-2y)(x+2y)=12$. Replace $x-2y$ in the second equation with 2 and we have a simple system of two equations:

$$\begin{aligned} x-2y &= 2 \\ x+2y &= 6 \end{aligned}$$

$(x,y)=(M,N)=(4,1)$ so $M+N=\boxed{5}$.

30. The area of a triangle given adjacent sides a and b and angle between C is $\frac{1}{2}ab\sin(C)$.

$$\begin{aligned} \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\sin(C) &= \frac{1}{5} \\ \sin(C) &= \frac{4}{5} \end{aligned}$$

$\cos(C)$ could be either $\boxed{\pm\frac{3}{5}}$ depending on whether the angle was acute or obtuse.

31. The figure shows an inscribed right triangle with leg coinciding with the side of the octagon. Draw the median to the hypotenuse to form an isosceles triangle with vertex angle 45°. This makes the larger angle of the right triangle $\frac{180-45}{2}^\circ = \frac{135}{2}^\circ$. Given this, we can infer that the height of triangle is $3\tan\left(\frac{135}{2}^\circ\right)$. Using the half-angle formula for tangent, we find that $\tan\left(\frac{135}{2}^\circ\right) = \frac{1-\cos(135^\circ)}{\sin(135^\circ)} = \frac{2+\sqrt{2}}{\sqrt{2}} = 1+\sqrt{2}$. The area of the triangle is $\frac{1}{2}\cdot 3\cdot 3(1+\sqrt{2}) = \boxed{\frac{9+9\sqrt{2}}{2}}$.

32. A is $3\ln(2) \approx 3\cdot .693 \approx 2.079$. B is $\sin\left(\frac{\pi}{7}\right) < \sin\left(\frac{\pi}{6}\right) < .5$. C is $\arctan\left(\frac{43}{25}\right) = \arctan(1.72) \approx \arctan(\sqrt{3}) \approx \frac{\pi}{3} \approx 1.05$. D is $\sqrt{3} \approx 1.732$. In ascending order of magnitude, we have $\boxed{\text{B, C, D, A}}$.

33. Having consumed 10 servings of pork rinds, he is at $10(9\cdot 9+18\cdot 4) = 1530$ calories. He can consume up to $1998-1530 = 468$ additional calories in the form of bacon, which has $3\cdot 9+3\cdot 4 = 39$ calories per slice. He can eat up to $468 \div 39 = \boxed{12}$ slices

34. Rewrite in standard form by completing the square:

$$\begin{aligned}
32x^2 - 128x + 9y^2 &= -56 \\
32[x^2 - 4x] + 9y^2 &= -56 \\
32[(x-2)^2 - 4] + 9y^2 &= -56 \\
32(x-2)^2 - 128 + 9y^2 &= -56 \\
32(x-2)^2 + 9y^2 &= 72 \\
\frac{(x-2)^2}{\frac{9}{4}} + \frac{y^2}{8} &= 1
\end{aligned}$$

We have an ellipse with $a = \frac{3}{2}$ and $b = 2\sqrt{2}$. The area is $\pi ab = \boxed{3\pi\sqrt{2}}$.

35. Let A be the large side of ΔS, B be the large side of ΔT, a be the small side of ΔS, and b be the small side of ΔT. We know that $A = 5+B$ and $b = a-2$. Divide through by B in the first equation and b in the second equation to obtain $\frac{A}{B} = \frac{5}{B}+1$ and $1 = \frac{a}{b} - \frac{2}{b}$. Add the equations together to obtain $\frac{A}{B}+1 = \frac{a}{b}+\frac{5}{B}-\frac{2}{b}+1$. Due to similarity $\frac{A}{B} = \frac{a}{b}$ so we can subtract this off both equations leaving $0 = \frac{5}{B} - \frac{2}{b}$ which can be rearranged to obtain $\frac{b}{B} = \boxed{\frac{2}{5}}$.

36. The volume of a frustum of a cone with small radius r, large radius R, and height h is $\frac{1}{3}\pi h(R^2+Rr+r^2)$. $\frac{1}{3}\pi\cdot 3(6^2+6\cdot 3+3^2) = \boxed{63\pi}$.

37. $8127 = 3^3\cdot 7\cdot 43$, so 8127 has $4\cdot 2\cdot 2 = 16$ total factors. The 15th number listed would be the next smallest factor after 8127 itself which is 8127 divided by its smallest factor. $8127 \div 3 = \boxed{2709}$.

38. Using Fermat's Little Theorem, we can gather that $2^{13} \equiv 2 \mod 13$.

$$\begin{aligned}
2^{73} &\mod 13 \equiv \\
\left(\left(2^{13}\right)^5 \cdot 2^8\right) &\mod 13 \equiv \\
\left(2^5 \cdot 2^8\right) &\mod 13 \equiv \\
2^{13} &\mod 13 \equiv \\
2 \cdot \left(2^6\right)^2 &\mod 13 \equiv \\
2 \cdot (-1)^2 &\mod 13 \equiv \\
\boxed{2} &\mod 13
\end{aligned}$$

39. The sum will be a string of 11 ones in base 2, plus 9 additional ones (since each number ended with 1). An n-digit string of ones in base 2 is equal to $2^n - 1$, so we have $2^{11} - 1 + 9 = 2^{11} + 8 = \boxed{2056}$.

40.

$$\begin{aligned}
\log_2(\sin(x)) &= \log_2(\sec(x)) - 1 \\
\log_2(\sin(x)) - \log_2(\sec(x)) &= -1 \\
\log_2(\sin(x)\cos(x)) &= -1 \\
\sin(x)\cos(x) &= \frac{1}{2} \\
2\sin(x)\cos(x) &= 1 \\
\sin(2x) &= 1
\end{aligned}$$

$\sin(2x) = 1$ whenever $2x = \frac{\pi}{2} + 2\pi n$ (n is any integer). In the interval, this gives us two possible values, $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. However, if $x = \frac{5\pi}{4}$, both $\sin(x)$ and $\sec(x)$ would be negative, which is disallowed in the domain of the $\log_2(x)$ function. This leaves $x = \boxed{\frac{\pi}{4}}$ as the only possible solution.