Mu Alpha Theta
2006 National Convention

Solutions
Analytic Geometry
Alpha Division

1. **A** Let r be the radius of the smaller circle, R be the radius of the larger circle. ½ of the tangent chord forms a right triangle with sides r, R, and 4. Our shaded area is equal to: $(R^2 - r^2)\pi = 4^2 \cdot \pi = 16\pi$
2. **C** 1/3 of the distance in the x direction is $\left(\frac{8-2}{3}\right) = 2$ and 1/3 of the distance along the y direction is $\left(\frac{7-3}{3}\right) = \frac{4}{3}$.

   Therefore, the point in question is: $\left(2+2, 7-\frac{4}{3}\right) = \left(4, \frac{17}{3}\right)$
3. **B** $r_{circum} = \frac{a \cdot b \cdot c}{4 \cdot Area} = 6.5 \quad r_{in} = \frac{Area}{semiperimeter} = \frac{30}{15} = 2 \quad \frac{r_{circum}}{r_{in}} = 3.25 = \frac{13}{4}$
4. **C** The dot product of the answer with both $\langle 1,2,3 \rangle$ and $\langle 4,5,6 \rangle$ must be zero. Only answer C fits this description
5. **D** This equation can be factored as $(2x - y + 1)^2 = 0$ which is a single line
6. **A** Choose a point on the second line, say $(6,2)$. The shortest distance from a point (x, y) to a line

   $Ax + By + C = 0$ is found by $\frac{|Ax + By + C|}{\sqrt{A^2 + B^2}} = \frac{39}{13} = 3$
7. **A** The numerator can be rewritten $4x - 3 = 4(x-1) + 4 - 3 = 4(x-1) + 1$. Therefore our hyperbola can be found by: $y = \frac{4(x-1)+1}{x-1} = 4 + \frac{1}{x-1}$ and $(y-4) = \frac{1}{x-1}$, which has center (1,4)
8. **A** $g(x)$ will look as if *f* was rotated about the line *y=x*, and *h(x)* resembles this *g(x)* rotated about the *x*-axis, which leaves *h(x)=-g(x)*.
9. **E** The value of the cosine of the angle in between the vectors **a** and **b** is equal to: $\frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|} = \frac{7}{25}$: 7 + 25 = 32.
10. **C** $\sin\left(\frac{3\pi}{2}\right) = -1$, $\cos\left(\frac{3\pi}{2}\right) = 0$ and det $\begin{vmatrix} -1 & -i & -1 \\ i & 0 & i \\ 1 & i & 1 \end{vmatrix} = 0$. (This is more evident since the first row is a multiple of the third row)
11. **D** $x = \frac{-b}{2a} = -4 \quad y = (-4)^2 + (8)(-4) - 7 = -23$
12. **B** The sphere is centered about the origin with radius 3. The point is located 5 units away from the origin, therefore the distance between the point and the sphere is 2.
13. **B** Each of the fifth roots of –2 lie on a circle around the origin with radius $r = \sqrt[5]{2}$, where the absolute value of each point on the circle is $\sqrt[5]{2}$. Therefore, the sum of 5 fifth roots is $5\sqrt[5]{2}$.
14. $h = d \sin 30 = 800 \cdot (0.5) = 400$
15. **E** with $f(x) = \frac{x}{3-x}$, substitute: $f(x) \Rightarrow x$ and $x \Rightarrow g(x)$ to solve for *g(x)*, the inverse, to get $g(x) = \frac{3x}{x+1}$

    $g(-2) + g(2) = 6 + 2 = 8$
16. **B** substituting $r^2 = x^2 + y^2$ after squaring both sides, and using $\sec(2\theta) = \frac{1}{\cos(2\theta)} = \frac{1}{\cos^2\theta - \sin^2\theta}$ with $\cos\theta = x/r, \sin\theta = y/r$, we get $x^2 - y^2 = 1$, where $a = 1, b = 1, c = \sqrt{2}$.

17. **C** Dividing our octagon into 8 triangles, each has base *s* and height $\frac{s}{2}\tan\left(\frac{135}{2}\right)$. $\tan\left(\frac{a}{2}\right) = \frac{\sin a}{1+\cos a}$,

$\tan\left(\frac{135}{2}\right) = \frac{\sqrt{2}}{2-\sqrt{2}} = \sqrt{2}+1$ so 8 of the triangles has area $2s^2(\sqrt{2}+1)$

18. $4\sin\theta\cos\theta = 2\sin(2\theta)$ For $r = a\sin(b\theta)$ where b is even, the number of petals, p = 2b.

19. **B** The volume of the solid described by 3 vectors (**a, b, c**) can be found by $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$. Therefore,

$\mathbf{b}\times\mathbf{c} = \langle 2,0,0\rangle$ and $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \langle 1,1,0\rangle\times\langle 2,0,0\rangle = 2$

20. **D** The two points described are the foci of the conic section. The fact that the positive difference remains constant implies this is a hyperbola. The center is ½ way between the foci (0,1), making *a*=2. *c*=3 is the distance between the center and a focus, and $c^2 = a^2 + b^2$, so $b^2 = 5$.

21. **B** The distance between the latera recta (the width of the rectangle in question) = 2*c*, where $c = \sqrt{a^2 - b^2} = 2$.

The height of the rectangle is the length of a latus rectum, $\frac{2b^2}{a} = \frac{2(9)}{\sqrt{13}}$. The area of the rectangle is the product,

$h\times w$.

22. **D** The eccentricity of all parabolas is 1.

23. **E** The slopes of the asymptotes are found as $\pm\frac{b}{a} = \pm\frac{\sqrt{45}}{\sqrt{5}} = \pm 3$

24. **C** The domain of $\log_x a$ does not allow for x to be negative; the function is asymptotic for small values of x approaching $x = 0$. As *x* increases, $x - \log_x 256$ gets infinitely large, yielding $y = 0$. Rearranging the denominator, $\log x^x - \log 256$ yields *x*=4 an asymptote.

25. **A** Using Green's Theorem for polygons:

$(0\cdot 4) + (1017\cdot 568) + (79\cdot 0) - (0\cdot 1017) - (4\cdot 79) - (568\cdot 0)$ all divided by 2 = 288670.

26. **D** This is a line that passes through the origin, therefore can be represented by $\theta = a$, where *a* is the arctangent of the slope.

27. **C** Start with the equation $(x-3)^2 + (y+1)^2 = 5^2$ and expand & rearrange to get **C**.

28. **D** for a hyperbola, $e = \frac{c}{a} = \frac{a^2+b^2}{a}$, and therefore $\frac{1+\frac{\sqrt{5}-1}{2}}{1} = \frac{1+\sqrt{5}}{2}$

29. **A** We can rearrange $\log y = t\log\sqrt{2}$ to become , $t = \log_2 y^2$, so $x = 2^t = y^2$ But $y > 0$, and $x > 0$, so $\sqrt{x} = y$

30. **B** The five platonic solids are: tetrahedron, cube, octahedron, dodecahedron, and icosahedron with sides of 4, 6, 8, 12, and 20, respectively.