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1. **[C]** L'Hopital's Rule cannot be applied here, because $\frac{1}{0}$ is not an indeterminate form. This is just the definition of the derivative. Change the y's to h's. $\lim_{h\to 0}\frac{\cos\left(\frac{3\pi}{2}+h\right)-\cos\left(\frac{3\pi}{2}\right)}{h}=f'\left(\frac{3\pi}{2}\right)$ when $f(x)=\cos(x)$, so the value is $f'\left(\frac{3\pi}{2}\right)=-\sin\left(\frac{3\pi}{2}\right)=1$.

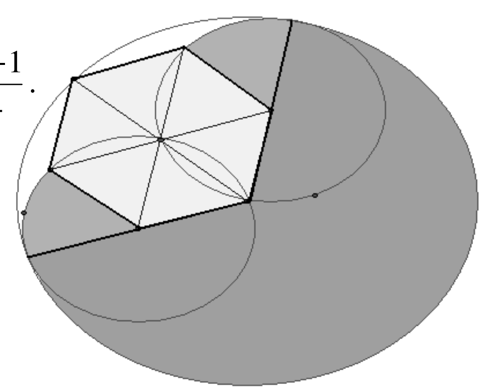
2. **[E]** Separating, $-\frac{1}{3y}dy=dt$. Integrating, $-\frac{1}{3}\ln|y|=t+C$. Substituting the initial condition, $-\frac{1}{3}(\ln 1)=0+C$, and $C=0$. So the final equation is $-\frac{1}{3}\ln|y|=t$. When $y=\frac{1}{3}$, $t=-\frac{1}{3}\ln\left(\frac{1}{3}\right)=\left(-\frac{1}{3}\right)(-\ln 3)=\frac{\ln 3}{3}$.

3. **[D]** $h(x)=x^{-\frac{4}{5}}$, and by the power rule, $h'(x)=-\frac{4}{5}x^{-\frac{9}{5}}$. $a+n=-0.8+-1.8=-2.6$.

4. **[A]** The slopes are $\pm\frac{\sqrt{20}}{\sqrt{5}}=\pm 2$, and the lines pass through the center, $(-1,3)$. Solving $y-3=2(x+1)$, $y=2x+5$, so $a=5$. The foci are $\sqrt{20+5}=5$ units from the center, and the transverse axis is vertical. So the foci are $(-1,8)$ and $(-1,-2)$, and $b=6$.

5. **[D]** $f(x)=\sec x$, so $f'(x)=\sec x\tan x$. By the product rule $f''(x)=\sec(x)\sec^2(x)+\tan(x)(\sec x\tan x)$, so $W=f''\left(\frac{\pi}{4}\right)=\sec^3\left(\frac{\pi}{4}\right)+\sec\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right)=2\sqrt{2}+\sqrt{2}=3\sqrt{2}$, and $W^2=18$.

6. **[A]** Let $u=x^4$, so $du=4x^3dx$. Substituting, $\int_0^1 x^3e^{x^4}dx=\frac{1}{4}\int_0^1 e^{x^4}\left(4x^3dx\right)=\frac{1}{4}\int_{0^4}^{1^4} e^u du$. Evaluating, $\frac{1}{4}\left[e^u\right]_0^1=\frac{e-1}{4}$.



7. **[D]** Since the area of a regular hexagon with side x is $1.5x^2\sqrt{3}$, solve to find that $x=6$. The portion of the large circle available is $\frac{240}{360}\left(\pi 12^2\right)=96\pi$. Then each of the two small circles has $\frac{60}{360}\left(\pi 6^2\right)=6\pi$ available. $96\pi+2(6\pi)=108\pi$.

8. **[D]** The velocity is increasing when the acceleration is positive. $v(t)=s'(t)=t^2-8t+15$, and $a(t)=v'(t)=2x-8$. Solving $2t-8>0$, find that $t>4$.

9. **[D]** The only possible values (from carrying ones) for *S*, *A*, and *M* are 9, 0, and 1. Then, from the last column, the only possible value of *Y* is 2. Digits remaining: $\{3,4,5,6,7,8\}$

$$\begin{array}{ccccccc} 1 & 1 & & & & 1 & \\ & & F & I & F & T & 2 \\ + & 9 & T & 1 & T & E & 9 \\ \hline 1 & 0 & E & R & I & C & 1 \end{array}$$

Note now that $F+T$ gives two different values in the two columns. That means that a one is carried into one of these columns. Because the column to the right of the first $F+T$ has a 1, and the 9 and 0 are already used, that means that the second $F+T$ column must have the one carried into it. Not also that because $F+T$ carries a 1 the first time, it must do the same the second time. Digits remaining: $\{3,4,5,6,7,8\}$

$$\begin{array}{ccccccc} 1 & 1 & & 1 & 1 & 1 & \\ & & F & I & F & T & 2 \\ + & 9 & T & 1 & T & E & 9 \\ \hline 1 & 0 & E & R & I & C & 1 \end{array}$$

Because the 0, 1, and 2 are used, the only possible combinations for *F* and *T* are (5,8), (6,7), (6,8), and (7,8). However, (5,8) and (6,7) are really not possible, because when *E*=3, there is no way the $T+E$ column can work. Let's check the other possibilities.

Checking $F=7, T=8$ – in this case, a contradiction occurs since *R* would be 8, which is already used.

$$\begin{array}{ccccccc} 1 & 1 & & 1 & 1 & 1 & \\ & & 7 & 6 & 7 & 8 & 2 \\ + & 9 & 8 & 1 & 8 & 5 & 9 \\ \hline 1 & 0 & 5 & R & 6 & 4 & 1 \end{array}$$

Checking $F=8, T=7$ – again, a contradiction occurs since *R* would be 8, which is already used.

$$\begin{array}{ccccccc} 1 & 1 & & 1 & 1 & 1 & \\ & & 8 & 6 & 8 & 7 & 2 \\ + & 9 & 7 & 1 & 7 & 5 & 9 \\ \hline 1 & 0 & 5 & R & 6 & 3 & 1 \end{array}$$

Checking (6,8) – in this case, *E*=4, and *T* must be 8, and then $C=3$. This is our answer!

$$\begin{array}{ccccccc} 1 & 1 & & 1 & 1 & 1 & \\ & & 6 & 5 & 6 & 8 & 2 \\ + & 9 & 8 & 1 & 8 & 4 & 9 \\ \hline 1 & 0 & 4 & 7 & 5 & 3 & 1 \end{array}$$

$F=6,\ I=5,\ T=8$, and the sum is 19.

10. [B] Plug in 0 for *y* to find the *x*-intercepts. Solving $x^2-4x=12$, the positive *x*-intercept is 6. Differentiating with respect to *y*, $2x+2yy'-4-6y'=0$. Substituting $(6,0)$, $12-4-6y'=0$, and $y'=\frac{4}{3}$.

11. [C] The probability that they are different colors is 1 – (prob of same colors). The probability of both being green is $\left(\frac{3}{9}\right)\left(\frac{2}{8}\right)=\frac{1}{12}$. The probability of both blue is $\left(\frac{2}{9}\right)\left(\frac{1}{8}\right)=\frac{1}{36}$. The probability of both red is $\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)=\frac{1}{6}$. $\frac{1}{12}+\frac{1}{36}+\frac{1}{6}=\frac{5}{18}$, and $1-\frac{5}{18}=\frac{13}{18}=\frac{52}{72}$.

12. [A] Use the rational root theorem with synthetic division to find that the only (repeating) roots are 2 and -1.

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13. [A] Consider the table.

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	3	6	12	24	17	29	22	34	27	39	32

Note that for $x > 5$, we alternate between the series 17, 22, 27… (for the even terms) and 29, 34, 39, … (for the odd terms).

Simply note that all values where x is divisible by 4 will end in a 2. Or you can go further and see that if x is even, every increase in 2 changes $f(x)$ by five, and model the situation for even $x > 4$ as $y-17=\frac{5}{2}(x-6)$, and find that $f(200)=502$.

14. [D] Choice A only works if the original quantity is negative. Choice B will not always work (e.g. if $|f(x)|<1$ for all values on the interval.) Choice C will not work if the original quantity is already nonnegative. Choice D will work, because all of the "negative area" (which must exist by the given conditions) will be treated as positive.

15. [D] The formula for the volume of a frustum is $V=\frac{\pi h}{3}\left(R^2+rR+r^2\right)$. R is a constant, 5. By similar cones (the "ghost cone" atop the frustum and the original cone), the relationship between r and h can be determined. $\frac{10-h}{10}=\frac{r}{5}$, and then $r=\frac{50-5h}{10}=5-\frac{h}{2}$. Substituting this into the volume formula, $V(h)=\frac{\pi h}{3}\left(25+5\left(5-\frac{h}{2}\right)+\left(5-\frac{h}{2}\right)^2\right)$. Simplifying,

$V(h)=\frac{\pi h}{3}\left(\frac{h^2}{4}-\frac{15h}{2}+75\right)=\frac{\pi}{3}\left(\frac{h^3}{4}-\frac{15h^2}{2}+75h\right)$. Differentiate with respect to time.

$\frac{dV}{dt}=\frac{\pi}{3}\left(\frac{3h^2}{4}-15h+75\right)\frac{dh}{dt}$. Evaluating at the requested instant,

$\frac{dV}{dt}=\frac{\pi}{3}\left(\frac{3(3)^2}{4}-15(3)+75\right)(2)=\frac{49\pi}{2}$.

16. [A] Let $u=x^4-3x^2-4$, and $du=4x^3-6x$. Substituting,

$$\int_0^1\frac{4x^3-6x}{x^4-3x^2-4}dx=\int_{0^4-3(0)^2-4}^{1^4-3(1)^2-4}\frac{1}{u}du=\int_{-4}^{-6}\frac{1}{u}du=\left[\ln|u|\right]_{-4}^{-6}=\ln 6-\ln 4=\ln\frac{6}{4}=\ln\frac{3}{2}.$$

17. [A] The derivative must change signs from negative to positive to guarantee a local maximum on the interval. Since f is twice-differentiable, f' is continuous, and since the function returns to its original value at $x=7$, and negative value of $f'(x)$, this guarantees that the derivative will undergo the appropriate sign change. Solving $-(4-k)-1<0$, find that $k<3$.

18. [D] The number of regions is $\frac{n(n+1)}{2}+1$, so $\frac{11(12)}{2}+1=67$.

19. [C] $g'(3)=\frac{1}{f'(2)}=2.5$

20. [C] Since $h(x)=\left(f(x)\right)^{-2}$, $h'(x)=-2\left(f(x)\right)^{-3}f'(x)$. $h'(1)=(-2)(8)(0.25)=-4$.

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21. **[C]** Note that $f(5-x) = f(-(x-5))$, which is the same as the graph of $f(x)$ being reflected about the *y*-axis, and then shifted right 5 units. Reflecting just the portion of the graph of $f(x)$ on the given interval and then translating it yields the same graph, but with a vertical reflection about the line $x = 2.5$. The graph's position relative to the *x*-axis remains unchanged, so the value is still 6.

22. **[C]** The Fundamental Theorem of Calculus cannot be applied here, because $f'(x)$ is not a continuous function. Since $f(x)$ is continuous, but not differentiable, at $x = 2$, then $f'(x) = \begin{cases} 2, & \text{if } x < 2 \\ 0, & \text{if } x > 2 \end{cases}$. Very easy to integrate these with geometry and the graphs. Since there is a jump discontinuity at *x*=2, $\int_0^3 f'(x)dx = \int_0^2 f'(x)dx + \int_2^3 f'(x)dx = 4 + 0 = 4$.

23. **[B]** $\sqrt{11-6\sqrt{2}} = 3 - 1\sqrt{2}$.

24. **[B]** Indeterminate form, so Use L'Hopital's Rule. $\lim_{x\to 0} \frac{1-\cos x}{x\sin x} = \lim_{x\to 0} \frac{\sin x}{x\cos x + \sin x} = \frac{0}{0}$. Still indeterminate, so use the rule again. $\lim_{x\to 0} \frac{\cos x}{(-x\sin x + \cos x) + \cos x} = \frac{1}{2}$.

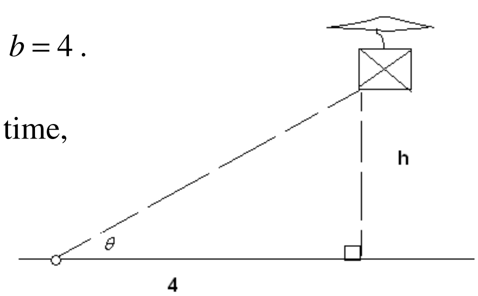
25. **[C]** Substituting, $x = y^2$, which is a parabola.

26. **[A]** The graphs intersect at (0,0) and (1,1). Integrating, $\int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

27. **[B]** There is one good combination out of ${}_5C_2 = \frac{5!}{2!3!} = 10$ possible combinations.

28. **[C]** Setting the derivative equal to the average rate of change, $2x - 3 = 1$, and solve to find $x = 2$.

29. **[C]** $f'(x) = 3x^2$, so lines normal to the graph have slope $\frac{-1}{3x^2}$. The slope of the line in question is $-\frac{1}{3}$, which occurs (on the positive side) at $x = 1$. So we wat the line to pass through the point on the graph $(1,1)$. Plugging in and solving for *b*, $1 = \frac{b-1}{3}$, find that $b = 4$.



30. **[D]** $\tan\theta = \frac{h}{4}$, so $4\tan\theta = h$. Differentiating with respect to time, $4\sec^2(\theta)\frac{d\theta}{dt} = \frac{dh}{dt}$. When the crate is 3km high, $\sec\theta = \frac{5}{4}$, and substituting, $4\left(\frac{5}{4}\right)^2 \frac{d\theta}{dt} = 30$. Solving, $\frac{d\theta}{dt} = -\frac{24}{5}$.
Since the question asks for the rate of decrease, the negative is unnecessary.