

*National MA © 2008 – Mu Limits and Derivatives Answers*

- 1) D
- 2) C
- 3) D
- 4) A
- 5) B
- 6) D
- 7) E
- 8) C
- 9) C
- 10) B
- 11) B
- 12) B
- 13) E
- 14) C
- 15) D
- 16) A
- 17) B
- 18) A
- 19) A
- 20) C
- 21) C
- 22) E
- 23) B
- 24) D
- 25) C
- 26) A
- 27) B
- 28) B
- 29) C
- 30) D

**National MA ⊗ 2008 – Mu Limits and Derivatives Solutions Page 1**

1)  $\lim_{x \rightarrow \infty} \frac{9x^3 - 3x^2 + 7x - 2007}{3x^3 + 6x + 2007} =$   
 $\lim_{x \rightarrow \infty} \frac{9 - \frac{3}{x} + \frac{7}{x^2} - \frac{2007}{x^3}}{3 + \frac{6}{x^2} + \frac{2007}{x^3}} = \frac{9 - 0 + 0 - 0}{3 + 0 + 0} = 3$  **D**

2)  $\lim_{t \rightarrow 0} \frac{f(t+1) - f(1)}{t} = f'(1) = 18(1) = 18$  **C**

3)  $y' = \frac{4x^2 \exp(x^2) - 2 \exp(x^2)}{4x^2} = \frac{\exp(x^2)(2x^2 - 1)}{2x^2}$

$y'' = \exp(x^2) \frac{2x^2(4x) - 4x(2x^2 - 1)}{4x^4} + \dots$   
 $+ 2x \exp(x^2) \frac{2x^2 - 1}{2x^2} = \frac{\exp(x^2)(2x^4 - x^2 + 1)}{x^3}$  **D**

4)  $A = \frac{1}{2}x(2x - x^2) \quad \frac{dA}{dx} = 2x - \frac{3}{2}x^2 = 0$

$x = 0$  (minimum area) or  $x = \frac{4}{3}$  (maximum area)

$A_{\max} = \frac{1}{2} \left( \frac{4}{3} \right) \left( 2 \left( \frac{4}{3} \right) - \left( \frac{4}{3} \right)^2 \right) = \frac{16}{27}$  **A**

5)  $h(x) = \sqrt{x + h(x)} \quad h(2) = \sqrt{2 + h(2)} \Rightarrow h(2) = 2$

$h'(x) = \frac{1 + h'(x)}{2\sqrt{x + h(x)}} \Rightarrow h'(2) = \frac{1 + h'(2)}{2\sqrt{2 + 2}}; h'(2) = \frac{1}{3}$  **B**

6)  $\lim_{x \rightarrow 1} h(x)$  exists if  $\lim_{x \rightarrow 1^+} h(x) = \lim_{x \rightarrow 1^-} h(x)$

$\lim_{x \rightarrow 1^+} h(x) = \lim_{x \rightarrow 1^+} x + 7 = 8$

$\lim_{x \rightarrow 1^-} h(x) = \lim_{x \rightarrow 1^-} 9 - x^2 = 8$

Since  $\lim_{x \rightarrow 1^+} h(x) = \lim_{x \rightarrow 1^-} h(x)$ , the limit exists

regardless of the value of  $a$ . **D**

7) Take derivative with respect to  $x$  on both sides

$12x^2 - 2xy^2 - 2x^2y \frac{dy}{dx} + 4y \frac{dy}{dx} = 0$

$\frac{dy}{dx} \Big|_{(-1,2)} = \frac{2xy^2 - 12x^2}{4y - 2x^2y} \Big|_{(-1,2)} = -5$  **E**

8)  $y' = 3x^2 - 4x + 1 \quad y' > 0$  for all  $x \in (1,4)$

$y$  is increasing on the interval  $(1,4)$

$y'' = 6x - 4 \quad y'' > 0$  for all  $x \in (1,4)$

$y$  is concave up on the interval  $(1,4)$  **C**

9) If  $a \geq 1$ ,  $a^n \rightarrow \infty$  AND  $n^a \rightarrow \infty$  divergent

If  $0 \leq a < 1$  then  $\lim_{n \rightarrow \infty} \frac{n^a}{a^{-n}} \equiv \lim_{n \rightarrow \infty} \frac{a \cdot n^{a-1}}{-\ln(a)a^{-n}} = 0$

If  $-1 < a < 0$ , let  $b = -a$

$\lim_{n \rightarrow \infty} (-b)^n n^{-b} = \lim_{n \rightarrow \infty} \frac{(-1)^n b^n}{n^b} = 0.$

If  $a = -1$ ,  $\lim_{n \rightarrow \infty} (-1)^n n^{-1} = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0.$

If  $a < -1$ , let  $b = -a$ :

$\lim_{n \rightarrow \infty} (-b)^n n^{-b} = \lim_{n \rightarrow \infty} \frac{(-1)^n b^n}{n^b} = \infty$  because  $b^n$

diverges more rapidly than  $n$  raised to any power. Thus the limit converges only when  $-1 \leq a < 1$ . **C**

10)  $r = \sqrt{(x-2)^2 + (y+5)^2} = \sqrt{(y^2+1)^2 + (y+5)^2}$

$\frac{dr}{dy} = \frac{2(y^2+1)(2y) + 2(y+5)}{2\sqrt{(y^2+1)^2 + (y+5)^2}} = 0$

Set numerator to zero:  $4y^3 + 6y + 10 = 0$

$y = -1 \Rightarrow r = \sqrt{20} = 2\sqrt{5}$  **B**

11)  $V = \frac{4\pi}{3} r^3 \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi$

$\frac{dV}{dt} \Big|_{r=2} = 4\pi(2)^2 \frac{dr}{dt} = 4\pi \Rightarrow \frac{dr}{dt} = \frac{1}{4}$

$A = 4\pi r^2 \quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(2) \left( \frac{1}{4} \right) = 4\pi$  **B**

12)  $\lim_{n \rightarrow \infty} \left( 1 + \frac{r}{n} \right)^{m+n} = e^{rt} \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n} \right)^{2n-3} = e^{-2}$  **B**

13)  $\ln h(x) = \frac{1}{x} \ln x \quad \frac{h'(x)}{h(x)} = \frac{1}{x^2} - \frac{1}{x^2} \ln x$

$h'(x) = \left( \frac{1}{x^2} - \frac{1}{x^2} \ln x \right) x^{1/x} = 0, \quad x^{1/x} \neq 0$  for any  $x$

$\frac{1}{x^2} - \frac{1}{x^2} \ln x = 0 \Rightarrow x = e \quad \text{Max} = e^{1/e}$  **E**

14) First derivative:  $n!x^{n!-1}$

Second derivative:  $n!(n!-1)x^{n!-2}$

Third derivative:  $n!(n!-1)(n!-2)x^{n!-3}$

$n^{\text{th}}$  derivative:

$n!(n!-1)(n!-2) \cdot \dots \cdot (n!-n+2)(n!-n+1)x^{n!-n}$

The coefficient equals  $\frac{(n!)!}{(n!-n)!}$  **C**

15)  $\left| \left( \frac{x}{3} + \pi - 1 \right) - \pi \right| < 0.3 \quad \frac{1}{3}|x-3| < 0.3$

**National MA @ 2008 – Mu Limits and Derivatives Solutions Page 2**

$|x-3| < 0.9$  Choose  $\delta = 0.9$  **D**

16)  $\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$   $r(\theta) = 1 - \cos(2\theta)$   
 $r'(\theta) = 2\sin(2\theta)$

$$\frac{dy}{dx} = \frac{\left(\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{\left(\sqrt{3}\right)\left(\frac{1}{2}\right) - \left(\frac{3}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = -3\sqrt{3} \text{ A}$$

17) I)  $g'(x)$  could be negative in between the integer values – False

II) There is a critical point at  $x = 1$  but not necessarily a minimum – False

III)  $g'(0) = 0$  – True

IV) Same reason as I – False **B**

18)  $\lim_{n \rightarrow \infty} \frac{1}{n} [e^{1/n} + e^{2/n} + \dots + e^{n/n}] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{k/n}$

This is a Riemann Integral approximation of

$$\int_0^1 e^x dx = e - 1 \text{ A}$$

19)  $g'(x) = 2(1 + f(x))f'(x)$

$$g''(x) = 2(1 + f(x))f''(x) + 2[f'(x)]^2$$

$$g''(0) = 2(1+1)(-1) + 2[0]^2 = -4 \text{ A}$$

20)  $F'(x) = \sin(x^2)$   $F''(x) = 2x \cos(x^2)$

A)  $F''(x) < 0$  concave down

B)  $F''(x) < 0$  concave down

C)  $F''(x) > 0$  concave up **C**

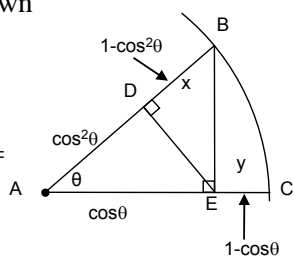
D)  $F''(x) < 0$  concave down

21)  $x = 1 - \cos^2 \theta$

$$y = 1 - \cos \theta$$

$$\lim_{\theta \rightarrow 0} \frac{x}{y} = \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{1 - \cos \theta} =$$

$$\lim_{\theta \rightarrow 0} (1 + \cos \theta) = 2 \text{ C}$$



22)  $\lim_{n \rightarrow \infty} \left(\frac{1}{5}\right)^n + \lim_{n \rightarrow \infty} \left(\frac{2}{5}\right)^n + \lim_{n \rightarrow \infty} \left(\frac{3}{5}\right)^n + \lim_{n \rightarrow \infty} \left(\frac{4}{5}\right)^n$

$$= 0 + 0 + 0 + 0 = 0 \text{ E}$$

23)  $f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = 2$ ,  $3c^2 - 1 = 2 \Rightarrow c = \pm 1$

$c$  cannot equal 1 since it's an endpoint. Sum of the values of  $c$  is  $-1$ . **B**

24)  $\lim_{x \rightarrow -1} \frac{2x^2 - 3x + 1}{x + 1} = \frac{2}{0}$  Does not exist **D**

25)  $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

$$\sqrt[3]{997} \approx \sqrt[3]{1000} + \frac{1}{3 \cdot 1000^{2/3}}(-3) = \frac{999}{100} \text{ C}$$

26) In order for  $\lim_{x \rightarrow c} f(x)$  to not exist and

$\lim_{x \rightarrow c} |f(x)| = L$  to exist, we must have

$$\lim_{x \rightarrow c^+} f(x) = - \lim_{x \rightarrow c^-} f(x) \text{ Let } f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

I)  $\lim_{x \rightarrow c^+} f(x) = 1$  &  $\lim_{x \rightarrow c^-} f(x) = -1$  both exist-True

II)  $\lim_{x \rightarrow c^+} |f(x) - 1| = 0$  and  $\lim_{x \rightarrow c^-} |f(x) - 1| = 2$  Thus

$\lim_{x \rightarrow c} |f(x) - L|$  does not exist-False

III)  $f(0)$  does exist-False

IV)  $\lim_{\Delta x \rightarrow 0^+} \frac{|f(0 + \Delta x) - f(0)|}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{1}{\Delta x}$  which

does not exist-False

Only I is true **A**

27)  $\lim_{x \rightarrow \infty} \sqrt{Ax^2 + 12x} - Bx = \lim_{x \rightarrow \infty} \frac{Ax^2 + 12x - B^2x^2}{\sqrt{Ax^2 + 12x} + Bx}$

For the limit to converge,  $A = B^2$ .

$$\lim_{x \rightarrow \infty} \frac{12x}{\sqrt{B^2x^2 + 12x} + Bx} = 3 \Rightarrow \frac{12}{\sqrt{B^2 + B}} = 3$$

$B = 2$  and  $A = 4$  **B**

28)  $g'(3) = \frac{1}{f'(g(3))}$   $f(x) = 3$  when  $x = 1$

$$g(3) = 1 \quad \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 1} = \frac{1}{4} \text{ B}$$

29) I)  $x \cdot |x| = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$  – YES

II)  $(x-1) \cdot |x| = x \cdot |x| - |x|$ ;  $x \cdot |x|$  works, but  $|x|$  is not differentiable at  $x = 0$  – NO

III)  $|x-1|^3 = \begin{cases} (1-x)^3 & x < 1 \\ (x-1)^3 & x \geq 1 \end{cases}$  – YES

IV)  $x^3 \cdot |x| = \begin{cases} -x^4 & x < 0 \\ x^4 & x \geq 0 \end{cases}$  – YES **C**

30)  $\lim_{x \rightarrow 0} \frac{x \sin(x)}{\exp(x^2) - 1} = \lim_{x \rightarrow 0} \frac{x \cos(x) + \sin(x)}{2x \exp(x^2)} =$

$$\lim_{x \rightarrow 0} \frac{-x \sin(x) + 2 \cos(x)}{4x^2 \exp(x^2) + 2 \exp(x^2)} = 1 \text{ D}$$