2009 Theta Ciphering

Solutions:

0. Find the number of permutations in the word Tennessee.

$\frac{9!}{4!\cdot 2!\cdot 2!} = 3780$

1. Find the equation of the line that is perpendicular to the line $y = 3x - 2$ and has the same x – intercept as the line $3x - 2y = 5$.

Solution: $m = -\frac{1}{3}$ and x – intercept is $\frac{5}{3}$ $\rightarrow y = -\frac{1}{3}x + b \rightarrow 0 = \left(-\frac{1}{3}\right)\left(\frac{5}{3}\right) + b \rightarrow b = \frac{5}{9}$

$y = -\frac{1}{3}x + \frac{5}{9}$ or $3x + 9y = 5$

2. Simplify: $(1 + i)^{2009} + (1 - i)^{2009}$

Solution: $((1 + i)^2)^{1004}(1 + i) = 2^{1004} + 2^{1004}i$; $((1 - i)^2)^{1004}(1 - i) = 2^{1004} - 2^{1004}i \rightarrow 2^{1005}$

3. Find the points of intersection of: $y = 3x^2 - 4x - 10$ and $2x - y = 1$.

Solution: $y = 2x - 1 = 3x^2 - 4x - 10 \rightarrow 3x^2 - 6x - 9 = 0, x^2 - 2x - 3 = 0 \quad (x - 3)(x + 1) = 0$
$x = 3, -1$; $(3, 5)$ and $(-1, -3)$

4. Solve: $\frac{\log_2 |(x+2)|}{\log_2 |(2x+3)|} = 2$

Solution: $\log_2(x + 2) = 2\log_2(2x + 3) \rightarrow (x + 2) = (2x + 3)^2 \rightarrow x + 2 = 4x^2 + 12x + 9$
$4x^2 + 11x + 7 = 0 \rightarrow (4x + 7)(x + 1) = 0 \rightarrow x = -1, -\frac{7}{4} \rightarrow x \neq -1$

5. A chemist has 2 solutions of acid; one is a 35% and the second is 44% acid. If he mixes the 2 solutions together to form a 40% acid solution how much of the 44% acid must he add to make a 240 ml solution?

Solution: x = 44% and $240 - x$ = 35% $\rightarrow .4(240) = .44x + .35(240 - x)$

$96 = .09x + 84$

$9x = 1200$

$x = \frac{400}{3}$ or $133\frac{1}{3}$

6. What is the largest prime number that will always divide a 6-digit number of the form *ababab*, where a and b are positive integers.

Solution: $a(100000) + b(10000) + a(1000) + b(100) + a(10) + b$

$a(101010) + b(10101) = (10101)(10a + b) \rightarrow 3\cdot 7\cdot 13\cdot 37$

7. Find the ordered pair (a, b) if : $2a + (3b + 1)i - (3b + 2) - 4ai = 3 - 3i$ (note: $i = \sqrt{-1}$)

   Solution: $2a - 3b - 2 = 3$ and $3b + 1 - 4a = -3 \rightarrow$ $\begin{matrix} 2a - 3b = 5 \\ -4a + 3b = -4 \end{matrix} \rightarrow a = -\frac{1}{2}; b = -2 \rightarrow \left(-\frac{1}{2}, -2\right)$

8. Find the area of the rectangle formed by the points of intersection of $x^2 + y^2 = 16$ and $25x^2 + 4y^2 = 100$.

   Solution: $\begin{matrix} 25x^2 + 4y^2 = 100 \\ -4x^2 - 4y^2 = -64 \end{matrix}$ $\rightarrow 21y^2 = 36 \rightarrow y = \pm\frac{6}{\sqrt{21}}$; $x^2 = \frac{100}{7} \rightarrow x = \pm\frac{10}{\sqrt{7}}$; Area = $\frac{80\sqrt{3}}{7}$

9. $f(x) = 4x - 3$; $g(f(x)) = 4x^2 - 5$; Find: $g(3)$

   Solution: $4x - 3 = 3 \rightarrow x = \frac{3}{2}$; $g\left(\frac{3}{2}\right) = 9 - 5 \rightarrow 4$

10. Find the value of the <u>determinant</u> of the product of : $\begin{bmatrix} 1 & -2 \\ 3 & -3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 & 0 \\ 0 & 2 & -4 \end{bmatrix}$

    Solution: $det\begin{bmatrix} -1 & 0 & 8 \\ -3 & 6 & 12 \\ 0 & 2 & -4 \end{bmatrix} \rightarrow -1(-24 - 24) + 0 + 8(-6) = 0$