2010 Nationals
Hustle Pre-Calc

Answers in Brief

1. $\cos 4020$
2. 1
3. 2
4. 15
5. 970
6. $17/25$
7. 4020 (or $4020+0i$)
8. $6i$ (or $0+6i$)
9. $4/3$
10. $-3\sqrt{5}$
11. 2010
12. 25
13. $4/5$
14. 5
15. 13
16. $-27/32$
17. $\pi/3$
18. 20,001
19. $-1/8$
20. 3
21. 5
22. 4
23. 10,101
24. 36π
25. 2

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① $(\cos(2)+i\sin(2))^{2010} = (\text{cis } 2)^{2010} = (e^{i\cdot 2})^{2010} = e^{4020i} =$

$\text{cis}(4020) = \cos 4020 + i\sin 4020$

Real part is $\boxed{\cos(4020)}$

② $\sum_{n=0}^{1005} x^{2n} = 0 \Rightarrow 1 + x^2 + x^4 + \ldots + x^{2010} = 0$

$a = \frac{0}{1}$ ← this would be coeff of x^{2009} (0 in this case)

$b = \frac{1}{1} \Rightarrow a+b = \boxed{1}$

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③ $\det(M) = \det(A)\cdot\det(O)$

$\det A = \ln 10 \cdot \log e - i^{-2010} \cdot i^{2012} = \frac{\log 10}{\log e}\cdot \log e - i^2 = 1 - -1 = 2$

$\det O = (\frac{1}{2})(4) - \frac{\log 2}{\log 14}\cdot\frac{\log 14}{\log 2} = 2 - 1 = 1$

$\Rightarrow \det M = 2\cdot 1 = \boxed{2}$

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(4) $\prod_{x=1}^{5} cis(x) = cis(1)\,cis(2)\,cis(3)\,cis(4)\,cis(5) = cis(n)$

by an extension of De Moivre's formula,

$cis(a) \cdot cis(b) = cis(a+b)$ (notice $cis(a) \cdot cis(a) = cis(2a) = cis^2(a)$)

So $cis(1) \ldots cis(5) = cis(15)$

$cis(15) = cis(n) \Rightarrow \boxed{n=15}$

(5) $\tan x + \cot x = 10$. Cube $(\tan x + \cot x)$ and we see $(\tan x + \cot x)^3 = \tan^3 x + \cot^3 x + 3\tan^2 x \cot x + 3 \tan \cot^2 x$

$10^3 = \tan^3 x + \cot^3 x + 3\tan x + 3\cot x$

$1000 = \tan^3 x + \cot^3 x + 3(10)$

$\Rightarrow \tan^3 x + \cot^3 x = \boxed{970}$

(6) $(\sin^2 x + \cos^2 x)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x = 1$

$\sin^4 x + \cos^4 x = 1 - 2\sin^2 x \cos^2 x$. We know $\sin 2x = \frac{4}{5}$

$\Rightarrow 2\sin x \cos x = 4/5$

$\Rightarrow \sin x \cos x = 2/5$

$\Rightarrow \sin^2 x \cos^2 x = 4/25$

$= 1 - 2\left(\frac{4}{25}\right)$

$= 1 - 8/25 = \boxed{17/25}$

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(7) $2010\sqrt{3} + 2010 \cdot i = 4020 \text{ cis } \pi/6$ Rotating $\pi/6$ clockwise

gives $4020 \text{ cis}(\pi/6 - \pi/6) = 4020 \text{ cis } 0$

$\boxed{= 4020}$ (also accept $4020 + 0 \cdot i$)

(8) $\sqrt{34}\, e^{\arctan\left(\frac{3}{5}\right) i} = \sqrt{34} \text{ cis}\left(\arctan \frac{3}{5}\right) = 5 + 3i$

$\sqrt{10}\, e^{\arctan \frac{1}{3} i} = \sqrt{10} \text{ cis}\left(\arctan \frac{1}{3}\right) = 3 + i$

Points on imaginary axis are in bi form, $b \in \mathbb{R}$

Setting distances equal, we have $\sqrt{(5-0)^2 + (3-b)^2} = \sqrt{(3-0)^2 + (1-b)^2}$

$\Rightarrow 25 + 9 - 6b + b^2 = 9 + 1 - 2b + b^2 \Rightarrow 24 = 4b \Rightarrow b = 6$

Point is $\boxed{6i}$ (or $0 + 6i$)

(9) $r = \dfrac{2}{1 - \frac{1}{2}\cos\theta}$ $\cos\theta = \dfrac{x}{r}$

$\Rightarrow r = \dfrac{2}{1 - \frac{1}{2}\frac{x}{r}} \Rightarrow r - \frac{1}{2}x = 2 \Rightarrow r = 2 + \frac{1}{2}x \Rightarrow 2r = 4 + x \Rightarrow 4r^2 = 16 + 8x + x^2 = 4x^2 + 4y^2$

$\Rightarrow 3x^2 - 8x + 4y^2 = 16 \Rightarrow x^2 - \frac{8}{3}x + \frac{4}{3}y^2 = \frac{16}{3} \Rightarrow x^2 - \frac{8}{3}x + \frac{64}{36} + \frac{4}{3}y^2 = \frac{16}{3} + \frac{64}{36}$

$\left(x - \frac{4}{3}\right)^2 + \frac{4}{3}y^2 = \frac{64}{9} \Rightarrow \dfrac{\left(x - \frac{4}{3}\right)^2}{64/9} + \dfrac{y^2}{16/3} = 1 \Rightarrow \text{Center: } \left(\frac{4}{3}, 0\right)$

$\Rightarrow \text{Sum} = \frac{4}{3} + 0 = \boxed{4/3}$

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(10) $x^2+3\sqrt{5}x-7i=0$ use quadratic formual!

$$x=\frac{-3\sqrt{5}\pm\sqrt{(3\sqrt{5})^2-4(1)(-7i)}}{2}=\frac{-3\sqrt{5}\pm\sqrt{45+28i}}{2}$$

$\sqrt{45+28i}=a+bi \Rightarrow 45+28i=a^2-b^2+2abi \Rightarrow a^2-b^2=45$
$ab=14$

Notice $a=7, b=2$ (alternatively, one could use cis notation)

$$\Rightarrow x=\frac{-3\sqrt{5}\pm(7+2i)}{2} \Rightarrow x=\frac{-3\sqrt{5}+7}{2}+i,\ x=\frac{-3\sqrt{5}-7}{2}-i$$

$$\frac{-3\sqrt{5}+7}{2}+\frac{-3\sqrt{5}-7}{2}=\boxed{-3\sqrt{5}}$$

(11) $f(x)=2010+xi$ $\quad |f(x)|=\sqrt{2010^2+x^2}$

$\min\limits_x \sqrt{2010^2+x^2} \Rightarrow$ w/o loss of generality, "ignore the square root part"

$\Rightarrow \min\limits_x 2010^2+x^2 \Rightarrow$ use quick vertex method

$x_v=\frac{-0}{2}=0$ So min is $\sqrt{2010^2}$ (when $x=0$)

Min value is $\boxed{2010}$

(12) Pure imaginary solns have the form $x=\pm r\cdot i$, $r\in\mathbb{R}$

$(ri)^4-(ri)^3+19(ri)^2-25(ri)-150=0 \Rightarrow r^4+r^3i-19r^2-25ri-150=0$

$\Rightarrow r^4-19r^2-150=0$ and $i(r^3-25r)=0$

$\Downarrow$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\Downarrow$

$(r^2+6)(r^2-25)=0$ and $i\cdot r\cdot(r-5)(r+5)=0$

$\Downarrow$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\Downarrow$

$r=\pm i\sqrt{6}, \pm 5$ $\quad\quad\quad\quad\quad r=0, \pm 5$

The pure imaginary solns are $x=\pm 5i$ So their product is $5i\cdot -5i=-25i^2=\boxed{25}$

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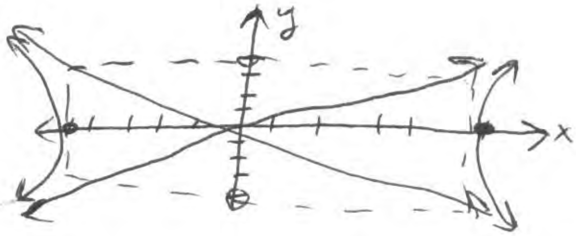
(13) $16x^2 - 25y^2 = 400 \Rightarrow \frac{x^2}{25} - \frac{y^2}{16} = 1 \Rightarrow$



Slope of asymptotes: $\pm \frac{b}{a} = \pm \frac{4}{5}$

$\tan \theta = m$ ← slope
angle inclination

$\tan\theta = \pm 4/5$

$\Rightarrow |\tan\theta| = |\pm 4/5| = \boxed{4/5}$

(14) $\sum_{n=4}^{\infty} \frac{5}{n^2-5n+6}$: $\frac{5}{n^2-5n+6} = \frac{A}{n-3} + \frac{B}{n-2} \Rightarrow A(n-2) + B(n-3) = 5$

$\Rightarrow A = 5, B = -5$

$\Rightarrow \sum_{n=4}^{\infty} \frac{5}{n^2-5n+6} = \sum_{n=4}^{\infty} \frac{5}{n-3} + \sum_{n=4}^{\infty} \frac{-5}{n-2}$

$= \frac{5}{1} + \cancel{\frac{5}{2}} + \cancel{\frac{5}{3}} + \cancel{\frac{5}{4}} + \dots + \cancel{\frac{-5}{2}} + \cancel{\frac{-5}{3}} + \cancel{\frac{-5}{4}} + \dots$

$= \boxed{5}$

(15) $\log_2(7!) = \log_2(5040) = x \Rightarrow 2^x = 5040$

$2^{10} = 1024,\ 2^{11} = 2048,\ 2^{12} = 4096,\ 2^{13} = 8192$

$\Rightarrow$ 13 is smallest integer lgr than $\log_2(7!)$

$\Rightarrow \boxed{13}$

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(16) $\cos x + \cos y = \frac{1}{2}$, $\sin x + \sin y = \frac{1}{4}$

Want: $\cos(x-y) = \cos x \cos y + \sin x \sin y$

Square our 2 first equations:

$\cos^2 x + 2\cos x \cos y + \cos^2 y = \frac{1}{4}$, $\sin^2 x + 2\sin x \sin y + \sin^2 y = 1/16$

Add equs:

$\cos^2 x + \sin^2 x + \sin^2 y + \cos^2 y + 2\cos x \cos y + 2\sin x \sin y = 5/16$

$\Rightarrow \quad 2(\cos x \cos y + \sin x \sin y) = -27/16$

$\Rightarrow \quad \cos(x-y) = \boxed{-27/32}$

(17) $\sin x + \sqrt{3}\cos x = 2\sin(x+\theta) = 2[\sin x \cos\theta + \sin\theta \cos x]$

$\Rightarrow 2\cos\theta \sin x = \sin x$ & $\sqrt{3}\cos x = 2\sin\theta \cos x$

$\Rightarrow \quad 2\cos\theta = 1$ & $\sqrt{3} = 2\sin\theta \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos\theta = \frac{1}{2}$ $\quad \Rightarrow \theta = \pi/3$

$\Rightarrow \theta = \frac{\pi}{3}$

$\boxed{\theta = \pi/3}$

(18) $\prod_{n=1}^{100} 100^{100} = \underbrace{100^{100} \cdot 100^{100} \cdots}\, 100^{100} = 100^{\sum_{n=1}^{100} 100} = 100^{100 \cdot 100}$

$= 100^{10,000} = 10^{20,000}$ $\quad (100 = 10^2)$

Numbers in the form 10^x have $x+1$ digits

So $\prod_{n=1}^{100} 100^{100}$ has $\boxed{20,001}$ digits

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(19) $\cos(2 \cdot \arcsin \frac{3}{4})$ let $\theta = \arcsin 3/4 \Rightarrow \sin\theta = 3/4$

$\hookrightarrow \cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = 1 - 2(3/4)^2$

$= 1 - 2 \cdot 9/16 = 1 - 18/16 = \boxed{-1/8}$

(20) $f(x) = \dfrac{x^3-8}{x^2-3x-4} = \dfrac{(x-2)(x^2+2x+4)}{(x+1)(x-4)} \Rightarrow x=-1$ & $x=4$ are vertical asymptotes

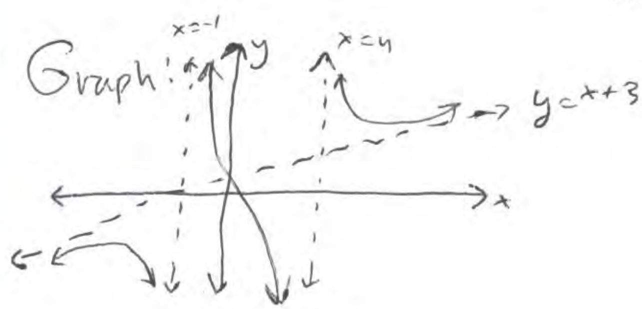
$\lim\limits_{x \to \infty} f(x)$ is not defined so there are no horizontal asymptotes.

Thus we divide the polynomials & see $\dfrac{x^3-8}{x^2-3x-4} = x+3+\dfrac{13x+4}{x^2-3x-4}$

$\Longrightarrow$ $y = x+3$ is a slanted asymptote

Thus we have $\boxed{3}$ total asymptotes.

Graph:



(21) $\tan(15°) = \tan(45° - 30°)$. Use addition angle formula for tangent to see $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ So $\tan(45-30) = \dfrac{\tan(45°) - \tan 30°}{1 + \tan 45° \tan 30°}$

$\Rightarrow \tan(15°) = \dfrac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \dfrac{3-\sqrt{3}}{3+\sqrt{3}} = \dfrac{3-\sqrt{3}}{3+\sqrt{3}}$ Multiply by conjugate of denom. to Rationalize

$\Longrightarrow \tan 15° = \dfrac{3-\sqrt{3}}{3+\sqrt{3}} \cdot \dfrac{3-\sqrt{3}}{3-\sqrt{3}} = \dfrac{9+3-6\sqrt{3}}{9-3} = \dfrac{12-6\sqrt{3}}{6} = 2 - \sqrt{3}$

$= a - \sqrt{b}$

$\Longrightarrow a=2, b=3 \Rightarrow a+b = \boxed{5}$

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(22) $\cos(\pi/8)$ Use $\cos(2\theta) = 2\cos^2\theta - 1$ and $2\theta = \alpha$ to see $\cos(\alpha) = 2\cos^2(\frac{\alpha}{2}) - 1$

$\Rightarrow \cos^2(\frac{\alpha}{2}) = \frac{\cos(\alpha)+1}{2}$

let $\alpha = \pi/4$ and $\theta = \pi/8$

$\Rightarrow \cos(\frac{\alpha}{2}) = \pm\sqrt{\frac{\cos(\alpha)+1}{2}}$

thus, $\cos(\theta) = \pm\sqrt{\frac{\cos\alpha + 1}{2}} = \cos(\frac{\pi}{8}) = \pm\sqrt{\frac{\cos\pi/4 + 1}{2}}$ $\Rightarrow \cos(\theta) = \pm\sqrt{\frac{\cos\alpha+1}{2}}$

$\cos(\pi/8) = \pm\sqrt{\frac{\sqrt{2}/2 + 1}{2}} = \pm\sqrt{\frac{\sqrt{2}+2}{4}} = \pm\frac{1}{2}\sqrt{\sqrt{2}+2}$. $\pi/8$ is in Quad 1 So $\cos(x)$ is positive

thus, $\cos(\pi/8) = \frac{1}{2}\sqrt{\sqrt{2}+2} = \frac{1}{2}\sqrt{\sqrt{a}+a} \Rightarrow a = 2$

$a^2 = \boxed{4}$

(23) 3, 7, 13, 21, 31, 43. Looking at differences b/w terms,

n=0	3	7	13	21	31	43
n=1	4	6	8	10	12	
n=2	2	2	2	2		
n=3	0	0	0			

implies this sequence is of second order, or that it is of the form

$an^2 + bn + c = x_n$ ← term value

So, $x_1 = 3 = a + b + c$ We now solve the system for a, b, c.

n=1 ↗ $x_2 = 7 = 4a + 2b + c$

$x_3 = 13 = 9a + 3b + c$ $\quad x_1 - x_2 = -4 = -3a - b = x_{12}$

$x_3 - x_1 = 10 = 8a + 2b = x_{31}$

Notice $2 \cdot x_{12} + x_{31} = 2 = 2a + 0 \cdot b \Rightarrow a = 2 - 1 = 1$

If $a = 1$, then $-4 = -3 - b \Rightarrow b = 1$ and we

have $3 = 1 + 1 + c \Rightarrow c = 1$. So $x_n = n^2 + n + 1$

We check with e.g. s.t. $x = 5$ $\quad x_5 = 5^2 + 5 + 1 = 31$ ✓

So x_n when $n = 100$ $\quad x_{100} = 100^2 + 100 + 1 = \boxed{10101}$

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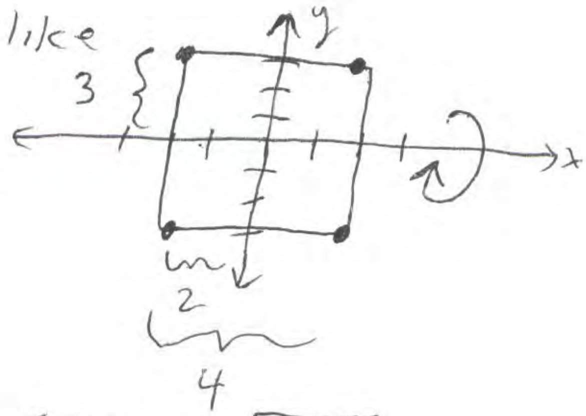
(24) $(2,3), (2,-3), (-2,3), (-2,-3)$ looks like



Notice when revolved around the x-axis we get a cylinder with radius 3 & height 4

The volume is thus $V = \pi(3^2)(4) = \boxed{36\pi}$

(25) $\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$

$\Rightarrow (2-\lambda)^2 - 1 = 0 \Rightarrow (2-\lambda)^2 = 1 \Rightarrow 2-\lambda = \pm 1$

$\Rightarrow 2-\lambda = 1$ & $2-\lambda = -1$

$\Downarrow$ & $\Downarrow$

$\lambda = 1$ & $\lambda = 3$

The answer is $|3-1| = \boxed{2}$