Mu Alpha Theta National Convention 2010 – Alpha Individual Solutions

1. C	6. D	11. C	16. A	21. A	26. A
2. A	7. A	12. A	17. B	22. C	27. C
3. B	8. A	13. B	18. D	23. B	28. B
4. A	9. B	14. E	19. D	24. C	29. B
5. B	10. E	15. D	20. C	25. A	30. D

The following were changed at the resolution center at the convention: 6 E, 19 E, 26 E.

1. $x^3-4x^2+5x-2=(x-1)(x-1)(x-2)$. The sum of the **distinct** real roots is thus $1+2=3$.
2. Each term in the expansion will be in the form $c\cdot x^i y^j z^k$. So to find the sum of the coefficients, we can let $x=y=z=1$. Therefore, the sum is $3^3\cdot(-1)^5=-27$.
3. The volume of the parallelepiped is the value of $\left|\langle 1,2,3\rangle\cdot\left(\langle 1,-2,4\rangle\times\langle 2,-2,3\rangle\right)\right|=18$.
4. We have
$$S_{100}=5000=S_{99}+100^2=S_{98}-99^2+100^2=S_0-1^2+2^2-3^2+\cdots+100^2.$$
$$=S_0+(2-1)(2+1)+(4-3)(4+3)+\cdots+(100-99)(100+99)$$
$$=S_0+1+2+3+\cdots+100=S_0+5050.$$
So $S_0=-50$. Then $S_{21}=S_0+1+2+\cdots+20-21^2=-50+210-441=-281$
5. $\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}=5$, so $\sqrt{x+5}=5$. So $x=20$.
6. $(a+bi)^2=a-bi\Rightarrow a^2-b^2+2abi=a-bi$. So $2ab=-b\Rightarrow a=-1/2$. It follows that $1/4-b^2=-1/2\Rightarrow b^2=3/4$. So $ab^2=-3/8$.
7. A matrix is invertible if its determinant is 0. The determinant of the matrix is a third degree polynomial with roots $3,\sqrt{2}$, and $-\sqrt{2}$.
8. We have 4 choices for the first digit, 5 choices for the next two digits, and 2 choices for the last digit. $4\cdot 5\cdot 5\cdot 2=200$.
9. This is an infinite geometric series with ratio $\cos\theta$. So the sum is equal to $A=\dfrac{1}{1-\cos^2\theta}=\dfrac{1}{\sin^2\theta}$. So $\sin\theta=1/\sqrt{A}$.
10. By the binomial theorem, each term will be of the form $\binom{10}{a}x^a\cdot\left(1/x^2\right)^{10-a}$. The x does not vanish for any value of *a*, so there is no constant term.
11. First we choose the 3 positions where 1, 2, and 3 will occur, in $\binom{6}{3}$ ways. The remaining positions can be filled in $4\cdot 3\cdot 2$ ways, so we have $\binom{6}{3}\cdot 4\cdot 3\cdot 2=\dfrac{6!}{3!}$ different orderings.

12. The amplitude of $f(x)=3\cos(\pi+x)+4\sin(\pi+x)$ is $\sqrt{3^2+4^2}=5$. The curve is shifted down 1, so the maximum value is 4.

13. Solving for $f^{-1}(x)$ in the equation $x=\dfrac{1+2f^{-1}(x)}{3f^{-1}(x)+4}$, we get $f^{-1}(x)=\dfrac{1-4x}{3x-2}$. Plugging in, we get $-3-(-1)=-2$.

14. With some algebra, we get $10^x=-4$, which has no real solutions.

15. $k^2=1-\sin 2x \Rightarrow \sin 2x=1-k^2 \Rightarrow \cos^2 2x=1-\sin^2 2x=1-(1-k^2)^2=2x^2-x^4$.

16. $y=\sqrt{9-x^2}$ is the top half of a circle centered at the origin with radius 3. $y=\dfrac{2}{3}\sqrt{9-x^2}$ is the top half of an ellipse centered at the origin with major axis 6 and minor axis 4. It follows that the area is the area of the semi-ellipse subtracted from the area of the semicircle, which is $\dfrac{1}{2}(9\pi-6\pi)=3\pi/2$.

17. $\log_{\sqrt{3}} 90=\dfrac{\log 3^2\cdot 5\cdot 2}{\frac{1}{2}\log 3}=\dfrac{2a+c+\frac{1}{2}b}{\frac{1}{2}a}=\dfrac{4a+b+2c}{a}$

18. The limit does not exist from the left.

19.

$$2\sin x=\sin 3x=\sin(x+2x)=\sin x\cos 2x+\cos x\sin 2x=\sin x(\cos^2 x-\sin^2 x)+2\cos^2 x\sin x=$$
$$=\sin x(\cos^2 x-1+\cos^2 x)+2\cos^2 x\sin x=\sin x(4\cos^2 x-1)\Rightarrow \cos^2 x=3/4$$

20. Let $a=e^x$. Then we have $a^3-8a^2+18a-10=0$. The product of the solutions is $a_1a_2a_3=10$. $x_1+x_2+x_3=\ln a_1+\ln a_2+\ln a_3=\ln 10$

21. Choose any point lying on the first line. (1,-1). Using the formula for distance from point to line, we get the distance to be: $\dfrac{3(1)+(-1)(-1)+4}{\sqrt{10}}=\dfrac{8}{\sqrt{10}}$.

22. $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}^{-1}=\dfrac{1}{\det}\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}=\dfrac{1}{-2}\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}$

23. The expression equals 0 if N is a multiple of 4. There are 24 multiples of 4 greater than 0 and less than 100.

24. A positive integer factor of $4200=2^3\cdot 3\cdot 5^2\cdot 7$ will be of the form $2^a 3^b 5^c 7^d$. There are 2 choices for a (2 or 3), 2 choices for b (0 or 1), 3 choices for c, (0,1, or 2), and 2 choices for d (0 or 1). So the number multiples of 4 is 24.

25. The cosine of angle between two vectors is their dot product divided by the product of their magnitudes: $17/\sqrt{396}$.

26. Divide each equation by xyz to get

$$\frac{3}{x}+\frac{1}{y}+\frac{1}{z}=8$$
$$\frac{1}{x}+\frac{2}{y}+\frac{2}{z}=6$$
$$\frac{2}{x}+\frac{4}{y}+\frac{3}{z}=13$$

Solving this system, we get $x=1/2$, $y=1/3$, $z=-1$.

27. $r=\sqrt{x^2+y^2}$, $r\sin\theta=y$. So $4\sqrt{x^2+y^2}=3-y \Rightarrow 16x^2+16y^2=9-6y+y^2$. It follows that the curve is an ellipse.

28. The elements of the *n*-th row consist of the coefficients of the expansion of $(1+x)^{n-1}$. So the sum of all entries is $2^0+2^1+\cdots+2^9=2^{10}-1=1023$.

29. The discriminant is equal to $b^2-4ac=16-4\cdot10\cdot2=-64$.

30. Such a progression is entirely determined by its first term and its common difference. If a progression has first term *a* and difference *d*, then $a+11d\le100$. When $d=1$, we have $a\le89$. When $d=2$, we have that $a\le88$. We see that *d* cannot be larger than 9. So the total number of progressions is: $\sum_{i=1}^{9}(100-11i)=900-11\cdot\frac{9\cdot10}{2}=900-495=405.$