Answers:

1. 10 – Number the bottles using binary digits. Assign each prisoner to one of the binary flags. Prisoners must take a sip from each bottle where their binary flag is set. With ten people there are 1024 unique combinations, so you could test up to 1024 bottles of wine.

2. 99 – The first man counts all the red hats he can see (Y) and then answers blue if the number is odd or red if the answer is even. Each subsequent wise man keeps track of the number of red hats known to have been saved from behind (X) and counts the number of red hats in front (Z). If Y was even and if X and Z are either both even or both odd, then the wise man would answer blue. Otherwise the wise man would answer red. If Y was odd and if X and Z are either both even or both odd, then the wise man would answer red. Otherwise the wise man would answer blue.

3. 3 – If you knew the fake coin was lighter, then the solution would have an easy explanation, but you don't know that. Number the coins 1 through 12. Weight coins 1, 2, 3, 4 against coins 5, 6, 7, 8. If they balance, then weight coins 9 and 10 against 11 and 8 (8 is a good coin). If the second weighing balances we know coin 12 is the counterfeit. The third weighing indicates whether it is heavy or light. If at the second weighing coins 11 and 8 are heavier than coins 9 and 10, either 11 is heavy or 9 is light or 10 is light. Weight 9 against 10. If they balance, 11 is heavy. If they don't balance, you know that either 9 or 10 is light. So the top coin is the fake. If at the second weighing coins 11 and 8 are lighter than coins 9 and 10, either 11 is light or 9 is heavy or 10 is heavy. Weight 9 against 10. If they balance, 11 is light. If they don't balance, you know that either 9 or 10 is heavy, so the bottom coin is the fake. Now if at the first weight the side with coins 5, 6, 7, 8 is heavier than the side with coins 1, 2, 3, 4, it means that either 1, 2, 3, 4 is light or 5, 6, 7, 8 is heavy. Weight 1, 2, 5 against 3, 6, 9. If they balance, it means that either 7 or 8 is heavy or 4 is light. By weighing 7 and 8 we obtain the answer, because if they balance, then 4 has to be light. If 7 and 8 do not balance, then the heavier coin is the counterfeit. If when you weigh 1, 2, 5 against 3, 6, 9 and the right side is heavier, then either 6 is heavy or 1 is light or 2 is light. By weighing 1 against 2, the solution is obtained. If the right side is lighter, then either 3 is light or 5 is heavy. By weighing 3 against a good coin the solution is easily determined. If at the first weighing coins 1, 2, 3, 4 are heavier than coins 5, 6, 7, 8, then repeat the previous steps.

4. 1349

5. 2519 – The solution for the answer is one less than the LCM of 10, 9, 8, 7, 6, 5, 4, 3, 2, and 1 since the LCM is the least number divisible by all of the numbers.

6. 5:23 am – Let the depth of snow at time t be t units. The speed of the plow at time t will be $1/t$. Define $t=0$ as the time it started snowing and $t=x$ the time the plow started.

The distance covered in the first hour is $\int_x^{x+1} 1/t\,dt = \ln\left(\frac{x+1}{x}\right)$. By the same reasoning the distance covered in the second hour is $\ln\left(\frac{x+2}{x+1}\right)$. Using the fact that the plow traveled twice as far in the first hour as the second, $\ln\left(\frac{x+1}{x}\right) = \ln\left(\frac{x+2}{x+1}\right)^2$. Solving this equation gives $x = \frac{\sqrt{5}-1}{2}$, which is the number of hours that elapsed between the time it started snowing and the snow plow left.

7. 10 – Let $x =$ the number of turns to reach the ant from the starting point, $y =$ the number of turns to reach ant from diagonal corner on same face, and $z =$ the number of turns to reach ant from an adjacent corner to ant. After one turn the spider will be on a diagonal corner of a common face as the ant, so the mean numbers of turns from the x position is one more than the mean number from the y position: $E(x) = 1 + E(y)$. Once at a y position there is a 2/3 chance it will then move to a z position, and a 1/3 chance back to an x position: $E(y) = \frac{2}{3}(1 + E(z)) + \frac{1}{3}(1 + E(x))$. If the spider arrives at a z position there is a 1/3 chance it will move to the ant and a 2/3 chance it will move back to a y position: $E(z) = \frac{1}{3}(1) + \frac{2}{3}(1 + E(y))$.

8. Fruit

9. Man: 52, Wife: 39 – M>W and M+W=91, and the man is now twice the age she was when he was her age. Some guess and check work gives the answer.

10. 1460 Sunset Boulevard – The only consecutive even integers that add to 8790 are 1460, 1462, 1464, 1466, 1468, and 1470, so the lowest house number is 1460.

11. 85% - There are only two combinations of 50 coins that add to $1: 40 P, 2 D, 8 N OR 45 P, 2 D, 1 N, 1 Q. The probabilities of choosing a penny for each scenario are 80% and 90%, so the probability is the average of the two: 85%.

12. 4 and 13 - xy can't be prime or the square of a prime since that would imply the two numbers are equal. P's statement implies that xy cannot have exactly two distinct proper factors whose sum is less than 100. Likewise, xy cannot be the cube of a prime. If S was sure that P could not deduce the numbers, then none of the possible summands of $x + y$ can be such that their product has exactly one pair of eligible factors. If P now knows that

$x+y$ is one of the values (11, 17, 23, 27, 29, 35, 37, 41, 47, 53), it enables P to deduce x and y. Of the eligible factorizations of xy, there must be precisely one for which the sum of the factors is in the list.

13. 1/2 – Either A throws more heads than B or A throws more tails than B, but not both. By symmetry, these two mutually exclusive possibilities occur with equal probability. Therefore the probability that A obtains more heads than B is 1/2. It is perhaps surprising that this probability is independent of the number of coins held by the players.

14. 20th – The probability $p(n)$ of getting a free ticket when you are the nth person in line is $p(n)=1\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\left(\frac{362}{365}\right)\cdots\left(\frac{365-(n-2)}{365}\right)\left(\frac{n-1}{365}\right)$, where $n\le 365$. We seek the least value of n such that $p(n)>p(n+1)\Rightarrow\frac{p(n)}{p(n+1)}>1$. $\frac{p(n)}{p(n+1)}=\frac{365}{366-n}\cdot\frac{n-1}{n}$, so we want that $365n-365>366n-n^2\Rightarrow n^2-n-365>0$. Solving this equation and rejecting the negative region, $n>\frac{1}{2}+\sqrt{365.25}\approx 19.1$, so 20th position suffices.

15. 30° – Let $x=\angle EDB$. Then $\angle BED=160^\circ-x$ and $\angle BDC=40^\circ$. Using the Law of Sines, $\frac{\sin x}{BE}=\frac{\sin(160^\circ-x)}{BD}=\frac{\sin(20^\circ+x)}{BD}$ and, because $\triangle BEC$ is isosceles, $\frac{\sin 80^\circ}{BD}=\frac{\sin 40^\circ}{BC}=\frac{\sin 40^\circ}{BE}$. Therefore, $\frac{\sin 80^\circ}{\sin 40^\circ}=\frac{BD}{BE}=\frac{\sin(20^\circ+x)}{\sin x}$, and using double-angle and product-to-sum formulas, $\sin(20^\circ+x)=2\cos 40^\circ\sin x=\sin(x-40^\circ)+\sin(x+40^\circ)$. Therefore, using the difference-to-product formula, $\sin(x+40^\circ)=\sin(20^\circ+x)-\sin(x-40^\circ)$ $=2\cos\left(\frac{(20^\circ+x)+(x-40^\circ)}{2}\right)\sin\left(\frac{(20^\circ+x)-(x-40^\circ)}{2}\right)=2\cos(x-10^\circ)\sin 30^\circ=\cos(x-10^\circ)$ $=\cos(10^\circ-x)=\sin(x+80^\circ)$. Therefore, $x+40^\circ=180^\circ-(x+80^\circ)\Rightarrow x=30^\circ$.

16. $24=\frac{6}{1-\frac{3}{4}}$

17. zero

18. K, M, and G

19. bit

20. 16th century

21. googol

22. slide rule

23. 21200

24. 12, 20, 4, 64

25. 1 – Since every integer in the group has a corresponding integer whose sum with the original number is 7, sums of seven can always be made. Since 50 is 1 more than a multiple of 7, choose 1 first, then whatever number the other player says, say the corresponding number and you will get to 50.

26. 56

27. 5

28. 1023 – The numbers are the powers of 2 from 1 to 512.

29. 35 – Use Pascal's Triangle to determine the number of ways.

30. 784

31. Mount Chimborazo in Ecuador

32. 84

33. Bookkeeper or Bookkeeping or variation

34. 1600 – Since the safe opens immediately on hitting the third number, it is not necessary to check the last number.

35. 97 – Each factor of 2 in the product creates another 0 at the end. The number of zeros at the end is $\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{8} \right\rfloor + \left\lfloor \frac{100}{16} \right\rfloor + \left\lfloor \frac{100}{32} \right\rfloor + \left\lfloor \frac{100}{64} \right\rfloor = 50 + 25 + 12 + 6 + 3 + 1 = 97$.

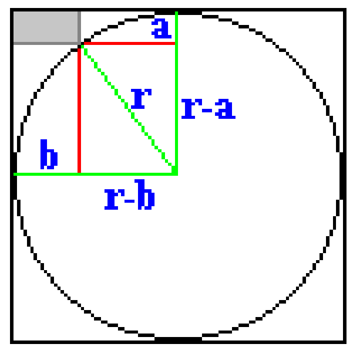
36.

Shooting order:	Survival of A	Survival of B	Survival of C
ABC	P(A,ABC) = 1/2	P(B,ABC) = 0	P(C,ABC) = ½
ACB	P(A,ACB) = 1/2	P(B,ACB) = 0	P(C,ACB) = ½
BAC	P(A,BAC) = 1/10	P(B,BAC) = 16/45	P(C,BAC) = 49/90
CAB	P(A,CAB) = 1/2	P(B,CAB) = 0	P(C,CAB) = ½
BCA	P(A,BCA) = 1/10	P(B,BCA) = 16/45	P(C,BCA) = 49/90
CBA	P(A,CBA) = 1/10	P(B,CBA) = 16/45	P(C,CBA) = 49/90
Total Survival chance:	27/90	16/90	47/90

37. 112 – Each child would get 16 coins.

38. Using Pythagorean Theorem,

$$(r-3)^2+(r-6)^2=r^2 \Rightarrow r=15$$



39. 11 – The bags should contain all the powers of 2 from 1 to 512, then the remaining 17 dollars in the last bag.

40. 10 – The square numbered lamps will be on (this is similar to the famous locker problem.

41. The number of recorded head injuries increased, but the number of deaths decreased.

42. They are equally contaminated – It doesn't matter how many transfers are made between the glasses or whether the contents are stirred. Provided that the volumes in the two glasses are equal, then any water in the water glass must be in the milk—there is nowhere else it can be. The milk that it has replaced must be in the water glass. The water glass, therefore, contains as much milk as the milk glass contains water.

43. The sheep are standing on the four corner points of an equal-sided tetrahedron. Three of the sheep are on an equilateral triangle on flat earth and the other is on a mound of earth in the center of the triangle.

44. The shepherd who had three loaves gets one coin and the shepherd who had five loaves should get seven coins – If there were eight loaves and three men, each man ate two and two-thirds loaves. So the first shepherd gave the hunter one third of a loaf and the second shepherd gave the hunter two and one-third loaves. The shepherd who gave one-third of a loaf should get one coin and the one who gave seven-thirds of a loaf should get seven coins.

45. It's the only number that contains all the numerals in alphabetical order.

46. 21978 (it's the only one)

47. 10% - Since 85% of the warriors lost an ear and 80% lost an eye (a total of 165%), at least 65% of them must have lost both an ear and an eye. Since at least 65% of the warriors lost both an ear and an eye, and 75% lost an arm (a total of 140%), at least 40% of them must have lost an ear, an eye, and an arm. Since at least 40% lost an ear, an eye, and an arm, and 70% lost a leg (a total of 110%), at least 10% of them must have lost one of each.

48. e – Contradictions between answers such as between c and d eliminate a as a possibility. B is false because if it were correct, then c would also be true. It follows that c is false because both a and b are false. Similarly, it follows that d is false. E is true which, of course, makes f false.

49. A pound of lead – Lead is weighed in the standard measure, in which 7000 grains equal a pound. Gold is measured in Troy weight, in which 5760 grains equal a pound.

50. 8439739020 – It is the product of 96420 and 87531.

51. 7 - $0 = 2b^2 + 5b - 133 = (b-7)(2b+19)$, so 7 is the only reasonable answer.

52. 40 km – Let S be the top of the lighthouse, A be a point on the horizon, and O be the center of the earth. Triangle OAS is a right triangle since SA is tangent to the earth. We thus have $\mathrm{OS}^2 = \mathrm{OA}^2 + \mathrm{AS}^2$, but $\mathrm{OA} = \dfrac{40000000}{2\pi}$ meters and $\mathrm{OS} = \mathrm{OA} + 125.7 = \mathrm{OA} + \dfrac{40\pi}{\mathrm{OS}^2}$ $\approx \mathrm{OA}^2 + 2\mathrm{OA} = 40\pi$. Therefore, $\mathrm{AS} = \sqrt{\mathrm{OS}^2 - \mathrm{OA}^2} \approx \sqrt{\dfrac{2(40\pi)(40000000)}{2\pi}} = 40000$ meters. Therefore, the horizon is approximately 40 km from the top of the lighthouse.

53. 22 – The probability that two people have different birthdays is 364/365. For n people, the probability is $p = \left(\dfrac{364}{365}\right)\left(\dfrac{363}{365}\right)\ldots\left(\dfrac{365-n+1}{365}\right)$. It can be checked that the product of these fractions is less than 1/2 when n changes from 22 to 23. (This is a well-known result and was also addressed on the first part of the Interschool Test.)

54. 5832/17496

55. Horizontal Line

56. Apothem

57. Paradox

58. Logarithms