Answers:

1. B
2. A
3. C
4. A
5. C
6. D
7. A
8. A
9. B
10. D
11. E
12. C
13. C
14. D
15. B
16. A
17. B
18. E
19. B
20. B
21. A
22. B
23. C
24. A
25. C
26. A
27. B
28. C
29. E
30. D

Solutions:

1. $\lim\limits_{k\to 4}\dfrac{k^2-2k-8}{k-4}=\lim\limits_{k\to 4}\dfrac{(k+2)(k-4)}{k-4}=\lim\limits_{k\to 4}(k+2)=4+2=6$

2. $k=\lim\limits_{x\to\infty}\dfrac{\left(1+\frac{1}{x}\right)^{x^2}}{e^x}\Rightarrow \ln k=\lim\limits_{x\to\infty}\ln\dfrac{\left(1+\frac{1}{x}\right)^{x^2}}{e^x}=\lim\limits_{x\to\infty}\left(x^2\ln\left(1+\dfrac{1}{x}\right)-x\right)$. Write

$x^2\ln\left(1+\dfrac{1}{x}\right)-x=\dfrac{\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}}{\frac{1}{x^2}}$ and use l'Hôpital's Rule to get

$\lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}}{\frac{1}{x^2}}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x+1}-\frac{1}{x}+\frac{1}{x^2}}{-\frac{2}{x^3}}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x^2(x+1)}}{-\frac{2}{x^3}}=\lim\limits_{x\to\infty}-\dfrac{x}{2(x+1)}=-\dfrac{1}{2}$, which

implies $k=e^{-1/2}=1/\sqrt{e}$.

3. $y=x^{x+y}\Rightarrow \ln y=(x+y)\ln x\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{x+y}{x}+\left(1+\dfrac{dy}{dx}\right)\ln x$, so at the point $(1,1)$,

$\dfrac{dy}{dx}=2+0=2$

4. Write $x^y=x^{x+x^y}=x^x x^{x^y}$. Since x and y are always positive, for $x<1$, x^y will be bounded above by 1 and is monotonically decreasing in x, so its limit must exist. Since $\lim\limits_{x\to 0^+}x^x=1$, by properties of limits we must have $\lim\limits_{x\to 0^+}x^y=\lim\limits_{x\to 0^+}x^{x^y}$. If the sought limit is called L, which we have already showed must exist, then L satisfies $L=\lim\limits_{x\to 0^+}x^L$; only $L=0$ will work.

5. The initial radius is $\dfrac{1+4\cdot 0}{1+0}=1$, so we are looking for the rate when the radius is 2, which occurs when $1+4t=2(1+t)\Rightarrow 2t=1\Rightarrow t=0.5$. The rate of increase of the area is $\dfrac{dA}{dt}=2\pi r\left.\dfrac{dr}{dt}\right|_{t=0.5}=2\pi\cdot 2\cdot\dfrac{4(1+0.5)-(1+4\cdot 0.5)}{(1+0.5)^2}=4\pi\cdot\dfrac{4}{3}=\dfrac{16\pi}{3}$.

6. Let $(a,4a-a^2)$ be the point of tangency. By using derivatives, the slope of the line through this point is $4-2a$. By using the definition of slope, the slope of the line

through this point is $\frac{4a-a^2-5}{a-2}$. Setting these equal, we must have

$$4-2a=\frac{4a-a^2-5}{a-2}\Rightarrow -2a^2+8a-8=4a-a^2-5\Rightarrow 0=a^2-4a+3=(a-1)(a-3)$$

$\Rightarrow a=1$ or $a=3$, and the slopes would be $4-2\cdot1=2$ or $4-2\cdot3=-2$, so the product of these slopes is $2(-2)=-4$.

7. $$k=\lim_{n\to\infty}\left(a^n-b^n\right)^{1/n}\Rightarrow \ln k=\lim_{n\to\infty}\frac{\ln\left(a^n-b^n\right)}{n}=\lim_{n\to\infty}\frac{a^n\ln a-b^n\ln b}{a^n-b^n}=\lim_{n\to\infty}\frac{\ln a-\left(\frac{b}{a}\right)^n\ln b}{1-\left(\frac{b}{a}\right)^n}$$

$$=\ln a\Rightarrow k=a$$

8. $$\lim_{x\to0}\frac{2\sin x-\sin 2x}{3\sin x-\sin 3x}=\lim_{x\to0}\frac{2\cos x-2\cos 2x}{3\cos x-3\cos 3x}=\lim_{x\to0}\frac{-2\sin x+4\sin 2x}{-3\sin x+9\sin 3x}$$

$$=\lim_{x\to0}\frac{-2\cos x+8\sin 2x}{-3\cos x+27\cos 3x}=\frac{-2+8}{-3+27}=\frac{6}{24}=\frac{1}{4}$$

9. Since the first and third entries in the first and second columns add up to the first and third entries in the third column, and since the determinant is 0, we must have

$$h(x)=f(x)+g(x)=\ln\left(e^x+1\right)+\ln\left(e^x-1\right)\Rightarrow h'(x)=\frac{e^x}{e^x+1}+\frac{e^x}{e^x-1}\Rightarrow h'(\ln 3)$$

$$=\frac{3}{3+1}+\frac{3}{3-1}=\frac{3}{4}+\frac{3}{2}=\frac{9}{4}$$

10. $$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}=\lim_{x\to a}\frac{\frac{2a^3-4x^3}{2\sqrt{2a^3x-x^4}}-a^{5/3}\cdot\frac{1}{3}x^{-2/3}}{0-a^{1/4}\cdot\frac{3}{4}x^{-1/4}}=\frac{-a-\frac{1}{3}a}{-\frac{3}{4}}=\frac{16a}{9}$$

11. $V=\pi r^2h+\frac{1}{2}\cdot\frac{4}{3}\pi r^3=\pi r^2h+\frac{2}{3}\pi r^3$ and $S=2\pi rh+\pi r^2+\frac{1}{2}\cdot4\pi r^2=2\pi rh+3\pi r^2$. When

$h=1.5$, $\frac{V}{S}=\frac{1.5r^2+\frac{2}{3}r^3}{3r+3r^2}=\frac{2}{9}r+\frac{5}{18}-\frac{5r/6}{3r+3r^2}$, so $\frac{b}{a}=\frac{5/18}{2/9}=\frac{5}{4}$

12. $$80\pi=2\pi rh+3\pi r^2\Rightarrow h=\frac{80-3r^2}{2r}\Rightarrow V=\pi r^2\left(\frac{80-3r^2}{2r}\right)+\frac{2}{3}\pi r^3=40\pi r-\frac{5}{6}\pi r^3\text{, so}$$

$V' = 40\pi - \frac{5}{2}\pi r^2$, which is equal to 0 when $r = 4$, and the derivative changes from positive to negative, indicating a maximum. Therefore, the maximum volume is $V = 40\pi(4) - \frac{5}{6}\pi(4)^3 = 160\pi - \frac{160}{3}\pi = \frac{320}{3}\pi$.

13. $f'(x) = 3x^2 - 6x - 45 = 3(x-5)(x+3) \Rightarrow f'(x) = 0$ when $x = 5$ or $x = -3$. $f(5) = 5^3$ $-3(5)^2 - 45(5) + 120 = -55$ and $f(-3) = (-3)^3 - 3(-3)^2 - 45(-3) + 120 = -27 - 27$ $+135 + 120 = 201 \Rightarrow a_1a_2 + b_1 + b_2 = 5(-3) - 55 + 201 = 131$

14. For appropriate values of x, $f(x) = \sum_{k=1}^{\infty}\left(1-\sqrt{x}\right)^k = \frac{1-\sqrt{x}}{1-\left(1-\sqrt{x}\right)} = \frac{1-\sqrt{x}}{\sqrt{x}} = x^{-1/2} - 1$, so

$f'(x) = -\frac{1}{2}x^{-3/2}$, meaning $\frac{f(x)}{f'(x)} = \frac{x^{-1/2}-1}{-\frac{1}{2}x^{-3/2}} = -2x + 2x^{3/2} \Rightarrow \left(\frac{f(x)}{f'(x)}\right)' = -2 + 3x^{1/2}$.

$\left(\frac{f(x)}{f'(x)}\right)' = 0$ when $x^{1/2} = \frac{2}{3} \Rightarrow x = \frac{4}{9}$, and $\left(\frac{f(x)}{f'(x)}\right)'$ changes from negative to positive at this value, meaning there is a minimum value at $x = \frac{4}{9}$. The minimum value would be $\frac{f\left(4/9\right)}{f'\left(4/9\right)} = -2\left(4/9\right) + 2\left(4/9\right)^{3/2} = -\frac{8}{9} + \frac{16}{27} = -\frac{8}{27}$.

15. The shade is made up of a rectangle with enclosed area 2 and a triangle above with point $(0,2)$ moving along the line $y = x + 2$. The base of the triangle is 2 and the height of the triangle is $y - 1$, where y is the coordinate of the variable point. Therefore, the enclosed area is $A = 2 + \frac{1}{2}\cdot 2\cdot(y-1) = y + 1$. Therefore, $\frac{dA}{dt} = \frac{dy}{dt}$. Since the point is moving along the line at a 45° angle at 2 units per second, the y-coordinate is changing at $\sqrt{2}$ units per second, and thus the area is changing at that rate as well.

16. $y\big|_{x=\pi/6} = \sin\frac{\pi}{6} = \frac{1}{2}$; $y'\big|_{x=\pi/6} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$; so the equation of the tangent is $y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$. Therefore, $\sin\frac{\pi}{8} \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{8} - \frac{\pi}{6}\right) = \frac{1}{2} - \frac{\pi\sqrt{3}}{48} = \frac{24 - \pi\sqrt{3}}{48}$.

17. First, solve for f and g. $f(x)=\dfrac{a}{x+f(x)}\Rightarrow\left(f(x)\right)^2+xf(x)-a=0$

$\Rightarrow f(x)=\dfrac{-x\pm\sqrt{x^2+4a}}{2}$, but choose $f(x)>0$ since x and a are both positive relevant to the limit, so $f(x)=\dfrac{-x+\sqrt{x^2+4a}}{2}$. Similarly, $g(x)=\dfrac{-x+\sqrt{x^2+4b}}{2}$.

Therefore, $\lim\limits_{x\to\infty}\dfrac{f(x)+\frac{x}{2}}{g(x)+\frac{x}{2}}=\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+4a}}{\sqrt{x^2+4b}}=1$.

18. By the last problem, $f(x)-g(x)=\dfrac{\sqrt{x^2+4a}-\sqrt{x^2+4b}}{2}\Rightarrow f'(x)-g'(x)$ $=\dfrac{1}{2}\left(\dfrac{x}{\sqrt{x^2+4a}}-\dfrac{x}{\sqrt{x^2+4b}}\right)$, which equals 0 only when $x=0$. Since $a>b$, this derivative changes from positive to negative at this value, making $x=0$ a local maximum. Because of the result of the last problem, $\lim\limits_{x\to\infty}\left(\sqrt{x^2+4a}-\sqrt{x^2+4b}\right)=0$, giving this graph a horizontal asymptote that is never reached. Therefore, the desired function has no minimum value.

19. If $x=1$, then $y+y^2+1+y=9\Rightarrow 0=y^2+2y-8=(y+4)(y-2)\Rightarrow y=-4$ or $y=2$. Using implicit differentiation, $x^3\dfrac{dy}{dx}+3x^2y+2xy\dfrac{dy}{dx}+y^2+1+\dfrac{dy}{dx}=0$, and plugging in $x=1$ gives $\dfrac{dy}{dx}+3y+2y\dfrac{dy}{dx}+y^2+1+\dfrac{dy}{dx}=0$, making $\dfrac{dy}{dx}=\dfrac{-y^2-3y-1}{2y+2}$. Plugging $y=-4$ into this expression gives $\dfrac{dy}{dx}=\dfrac{5}{6}$, and plugging in $y=2$ gives $\dfrac{dy}{dx}=-\dfrac{11}{6}$. Thus, $\dfrac{5}{6}-\dfrac{11}{6}=-\dfrac{6}{6}=-1$.

20. Dividing the numerator and denominator by $x-c$ makes them difference quotients, and since f and g are differentiable, taking limits as $x\to c$ are the limit definitions for derivatives of f and g.

21. Take a limit as $x\to c$, then use the squeeze theorem to squeeze ξ toward c and therefore show that $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$.

22. The proof from the previous question used the key assumption of continuity, which came from replacing $f(c)$ and $g(c)$ with the corresponding limits. It doesn't make sense to talk of continuity, however, when a function escapes to infinity. So we can't use a continuity argument which leads to the mean value theorem and squeeze theorem (although we do wind up there anyway by other arguments).

23. $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x^2}=\lim\limits_{x\to 0}\dfrac{2\cos x-2\cos 2x}{2x}=\lim\limits_{x\to 0}\dfrac{-2\sin x+4\sin 2x}{2}=\dfrac{0+0}{2}=0$

24. With the given information, since $V=\dfrac{4}{3}\pi r^3$, $\dfrac{dV}{dt}=4\pi r^2\dfrac{dr}{dt}\Rightarrow 3=4\pi(10)^2\dfrac{dr}{dt}$ $\Rightarrow\dfrac{dr}{dt}=\dfrac{3}{400\pi}$. Now, since $A=4\pi r^2$, $\dfrac{dA}{dt}=8\pi r\dfrac{dr}{dt}\Rightarrow\dfrac{dA}{dt}=8\pi(10)\left(\dfrac{3}{400\pi}\right)=\dfrac{3}{5}$.

25. $\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{\sin^2 x}=\lim\limits_{x\to 0}\dfrac{\cos x-\sec^2 x}{\sin 2x}=\lim\limits_{x\to 0}\dfrac{-\sin x-2\sec^2 x\tan x}{2\cos 2x}=\dfrac{0-0}{2}=0$

26. $f'(x)=\dfrac{(2x+3)(-1)-(4-x)2}{(2x+3)^2}=-\dfrac{11}{(2x+3)^2}$, so $f(x)+f'(x)=\dfrac{4-x}{2x+3}-\dfrac{11}{(2x+3)^2}$ $=\dfrac{-2x^2+5x+1}{(2x+3)^2}$. Its derivative is $\dfrac{(2x+3)^2(-4x+5)-4(-2x^2+5x+1)(2x+3)}{(2x+3)^4}$ $=\dfrac{-22x+11}{(2x+3)^3}$, which equals 0 when $x=1/2$, and the derivative changes from positive to negative at this point, creating a local maximum, and it's the only maximum, making is an absolute maximum. The value is $f\left(1/2\right)+f'\left(1/2\right)$ $=\dfrac{4-1/2}{2\left(1/2\right)+3}-\dfrac{11}{\left(2\left(1/2\right)+3\right)^2}=\dfrac{7}{8}-\dfrac{11}{16}=\dfrac{3}{16}$.

27. The positive terms in the sequence approach 1 from above, and the negative terms in the sequence approach -1 from above. Therefore, $(\liminf a_n)+(\limsup a_n)$ $=-1+1=0$.

28. The graph of f is a portion of the parabola whose largest value is approaching the point where the point with $x=-1$ should have been, which is $2(-1)^2-(-1)=3$; this would be the supremum.

29. I is false; for example, let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, then $\limsup a_n = 1$ and $\limsup b_n = 1$, but $a_n + b_n = 0 \Rightarrow \limsup(a_n + b_n) = 0 \neq 1 + 1$. II is false; if $a_n \to \infty$ and is monotonically increasing, then we would have $\limsup a_n = \liminf a_n$, but a_n doesn't converge since it approaches ∞ (this statement would be true is we knew that a_n was bounded). III is true; if the sequence takes on its minimum value for some $k < n$ then we would have strict inequality, and if the sequence takes on its minimum value for arbitrarily high n (one example would be the case where it's monotonically decreasing), then we would have equality.

30. First note that $\sin n$ only equals 0 at integer multiples of π, so because n is an integer, $\sin n$ never equals 0. Thus there is a constant $\varepsilon > 0$ such that for all integers n, $|\sin n| > \varepsilon$. Similarly, we know that $|\sin n| < 1$ for all integers n, so we can write $\liminf \varepsilon^{1/n} \leq \liminf |\sin n|^{1/n} \leq \liminf 1^{1/n}$. However, $1^{1/n} = 1$, making its limit inferior 1, and $\varepsilon^{1/n}$ is monotonically increasing and converges to $\varepsilon^0 = 1$, so its limit inferior is 1 also. Hence, by the squeeze theorem, we must also have $\liminf |\sin n|^{1/n} = 1$.