Answers:

1. B
2. D
3. A
4. B
5. C
6. A
7. D
8. A
9. D
10. B
11. D
12. C
13. E
14. B
15. A
16. B
17. D
18. D
19. B
20. C
21. B
22. A
23. C
24. E
25. B
26. E
27. B
28. B
29. B
30. D

Solutions:

1. $e^x y' - x = y' \Rightarrow y'\left(e^x - 1\right) = e^x y' - y' = x \Rightarrow y' = \dfrac{x}{e^x - 1}$

2. $z' - xz = -x \Rightarrow \dfrac{dz}{dx} = xz - x = x(z-1) \Rightarrow \int \dfrac{dz}{z-1} = \int x dx \Rightarrow \ln|z-1| = \dfrac{1}{2}x^2 + c$

$\Rightarrow z = Ce^{x^2/2} + 1 \Rightarrow z = 3e^{x^2/2} + 1$ since $z(0) = 4$

3. $x^2 + y^2 = c^2 \Rightarrow 2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y} \Rightarrow$ for orthogonal trajectories, $\dfrac{dy}{dx} = \dfrac{y}{x}$

$\Rightarrow \int \dfrac{dy}{y} = \int \dfrac{dx}{x} \Rightarrow \ln|y| = \ln|x| + c \Rightarrow y = kx$

4. Multiplying both sides of $2xydx + y^2 dy = 0$ by $1/y$ would create two terms, each of which features only one variable. Therefore, the integrating factor is $1/y$.

5. The set is linearly dependent if there is a set of real numbers c_1, c_2, and c_3, not all 0, such that $c_1(1-cx) + c_2(1+x) + c_3(2-6x) = 0 \Rightarrow (c_1 + c_2 + 2c_3) + (-c_1c + c_2 - 6c_3)x = 0$ for all x. This occurs if both numbers in parentheses are 0. Since there would then be two equations in three unknowns, there are an infinite number of values of c_1, c_2, and c_3 that would work. Since $c = \dfrac{c_2 - 6c_3}{c_1}$, c can be any real value as well.

6. Using the form $\dfrac{dQ}{dt} = (\text{rate of salt in}) - (\text{rate of salt out})$, we know that no salt is going in and $\left(\dfrac{Q \text{ pounds}}{100 \text{ gallons}}\right)\left(\dfrac{5 \text{ gallons}}{\text{minute}}\right) = \dfrac{Q}{20}\ \dfrac{\text{pounds}}{\text{minute}}$ is the rate out, so we are trying to solve $\dfrac{dQ}{dt} = -\dfrac{Q}{20} \Rightarrow \int \dfrac{dQ}{Q} = -\int \dfrac{dt}{20} \Rightarrow \ln|Q| = -\dfrac{t}{20} + c \Rightarrow Q = Ce^{-t/20}$. Using the fact that $Q(0) = 20$, we get that $Q = 20e^{-t/20}$.

7. $y(0.1) \approx 1 + 0.1(2\cdot 0 + 2\cdot 1) = 1.2$ and $y(0.2) \approx 1.2 + 0.1(2\cdot 0.1 + 2\cdot 1.2) = 1.46$

8. Because $v(1)=0$ and $s(1)=0$, the velocity function is $v(t)=\frac{(t-1)^3}{3}$ and the position function is $s(t)=\frac{(t-1)^4}{12}$, thus making $s(3)=\frac{(3-1)^4}{12}=\frac{16}{12}=\frac{4}{3}$.

9. $a(t)=-32\Rightarrow v(t)=-32t+c\Rightarrow v(t)=-32t+256$ since the initial velocity of the ball was 256 feet per second. The velocity will be positive when $t<8$ and negative when $t>8$, meaning the maximum occurs when $t=8$. Additionally, the height function would be $h(t)=-16t^2+256t$, meaning the height at $t=8$ would be $h(8)=-16(8)^2+256(8)=-1024+2048=1024$.

10. $P=P_0e^{rt}$, so to get the principal to double in 6 years, we must have $e^{6r}=2$ $\Rightarrow 6r=\ln 2\Rightarrow r=\frac{\ln 2}{6}$. Written as a percentage, this would be $\left(\frac{50\ln 2}{3}\right)\%$.

11. The characteristic equation is $0=m^2-m-2=(m-2)(m+1)\Rightarrow m=2$ or $m=-1$, so the two general solutions are e^{2x} and e^{-x}, meaning the general solution to the differential equation is $y=c_1e^{-x}+c_2e^{2x}$.

12. An ordinary differential equation only has one input variable, which A, B, and D have. Only C has functions of two different variables.

13. With this being a Bernoulli equation, making the substitutions $y=z^{3/2}$ and $y'=\frac{3}{2}z^{1/2}z'$ yields the equation $\frac{3}{2}z^{1/2}z'-\frac{3}{x}z^{3/2}=x^4z^{1/2}\Rightarrow z'-\frac{2}{x}z=\frac{2}{3}x^4$. The integrating factor for this new equation is $e^{-\int\frac{2}{x}dx}=\frac{1}{x^2}$, so multiplying both sides of the equation by that yields $\frac{1}{x^2}z'-\frac{2}{x^3}z=\frac{2}{3}x^2\Rightarrow\left(\frac{1}{x^2}z\right)'=\frac{2}{3}x^2\Rightarrow\frac{1}{x^2}z=\frac{2}{9}x^3+c$ $\Rightarrow z=cx^2+\frac{2}{9}x^5$. Since $y=z^{3/2}$, $y=\left(cx^2+\frac{2}{9}x^5\right)^{3/2}$.

14. $ydx+xdy=0\Rightarrow\int\frac{dy}{y}=-\int\frac{dx}{x}\Rightarrow\ln|y|=-\ln|x|+c\Rightarrow y=\frac{C}{x}$, which is a family of hyperbolas

15. The characteristic equation for this differential equation is $0=100m^2-20m+1$

$=(10m-1)^2 \Rightarrow m = 1/10$, which is a double root. Therefore, the solution to the differential equation is $N = c_1 e^{t/10} + c_2 t e^{t/10}$.

16. The order is the largest number of a derivative appearing in the equation, which in this case is 2.

17. The characteristic equation would be $(m-2)(m-6)(m-8) = m^3 - 16m^2 + 76m - 96$ $= 0$, so $y''' - 16y'' + 76y' - 96y = 0$ would be the corresponding differential equation.

18. Because this is a homogeneous differential equation, making the substitutions $y = vx$ and $y' = v + v'x$ yields $v + v'x = \dfrac{2v^4x^4 + x^4}{x \cdot v^3x^3} = \dfrac{2v^4+1}{v^3} \Rightarrow v'x = \dfrac{v^4+1}{v^3}$

$$\Rightarrow \int \frac{v^3}{v^4+1}dv = \int \frac{dx}{x} \Rightarrow \frac{1}{4}\ln\left|v^4+1\right| = \ln|x| + c \Rightarrow v^4 + 1 = kx^4 \Rightarrow \left(\frac{y}{x}\right)^4 + 1 = kx^4$$

$$\Rightarrow y^4 + x^4 = kx^8 \Rightarrow y^4 = kx^8 - x^4.$$

19. This is the solution for the general linear differential equation given as answer B.

20. This is an exact differential equation. Taking the antiderivative of $x + \sin y$ with respect to x yields $\dfrac{x^2}{2} + x\sin y + g(y)$ for some g. Taking the derivative of this with respect to y yields $x\cos y + g'(y)$, and this must equal $x\cos y - 2y$, which would make $g'(y) = -2y \Rightarrow g(y) = -y^2$. Therefore, $\dfrac{x^2}{2} + x\sin y - y^2 = c$ is the solution.

21. $P = P_0 e^{kt}$, so to get the population to double in 2 years, we must have $e^{2r} = 2$ $\Rightarrow 2r = \ln 2 \Rightarrow r = \dfrac{\ln 2}{2}$, which is the relative growth rate.

22. The characteristic equation is $0 = m^3 - 6m^2 + 11m - 6 = (m-1)(m-2)(m-3)$ $\Rightarrow m = 1,\ m = 2,$ or $m = 3$, so the three general solutions are e^x, e^{2x}, and e^{3x}, meaning the general solution to the differential equation is $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$.

23. $y' = 3x^2 + x \Rightarrow y = x^3 + \dfrac{x^2}{2} + c$

24. Linear differential equations have the form $b_n(x)y^{(n)}+b_{n-1}(x)y^{(n-1)}+...+b_1(x)y'$ $+b_0(x)y=g(x)$, and none of the given answer choices fit this form, so none of them are linear.

25. Using Newton's Law of Cooling, the differential equation for this scenario is $\frac{dy}{dt}=k(y-0)=ky\Rightarrow y=Ce^{kt}$, where y is the temperature of the bar. Since the initial temperature of the bar was 100°F, the equation is $y=100e^{kt}$. Additionally, $50=100e^{k\cdot 20}\Rightarrow 20k=\ln 0.5\Rightarrow k=\frac{\ln 0.5}{20}$, so the equation is $y=100e^{\left(\frac{\ln 0.5}{20}\right)t}$. Plugging in $y=25$ yields $25=100e^{\left(\frac{\ln 0.5}{20}\right)t}\Rightarrow\left(\frac{\ln 0.5}{20}\right)t=\ln 0.25\Rightarrow t=\frac{20\ln 0.25}{\ln 0.5}=40$, so it will take 40 minutes to reach that temperature.

26. Since $y''-y=0$, you can add any number of e^x and e^{-x}, and one of the answers must be the given answer, so the general form for the solution to this differential equation is $y=-x^2-2+c_1e^x+c_2e^{-x}$, which is not equivalent to any of the given answer choices.

27. $1=y(0)=c_1e^0+c_2e^0+4\sin 0=c_1+c_2$, and since $y'(x)=c_1e^x-c_2e^{-x}+4\cos x$, $-1=y'(0)=c_1e^0-c_2e^0+4\cos 0=c_1-c_2+4\Rightarrow c_1-c_2=-5$. Solving this simultaneous system yields $c_1=-2$ and $c_2=3$, making their product $-2\cdot 3=-6$.

28. Four days is four full half-lives, so the remaining amount of the substance would be $16\left(\frac{1}{2}\right)^4=16\cdot\frac{1}{16}=1$ gram.

29. A homogeneous differential equation $y'=F(x,y)$ would satisfy $F(tx,ty)=F(x,y)$. Because all quantities are raised to the fourth power, this would work for answer B. For answer A, $F(tx,ty)=t^2F(x,y)$; for answer C, $F(tx,ty)=\frac{1}{t}F(x,y)$; and for answer D, $F(tx,ty)=tF(x,y)$.

30. $y'=\frac{x+1}{y^4+1}\Rightarrow\int(y^4+1)dy=\int(x+1)dx\Rightarrow\frac{y^5}{5}+y=\frac{x^2}{2}+x+c$

$$\Rightarrow\frac{y^5}{5}+y-\frac{x^2}{2}-x=c$$