Answers:

1. C
2. A
3. B
4. B
5. D
6. C
7. A
8. A
9. B
10. D
11. C
12. E
13. C
14. D
15. B
16. B
17. C
18. A
19. B
20. B
21. C
22. E
23. D
24. D
25. B
26. C
27. A
28. B
29. D
30. D

Solutions:

1. $\sum_{n=1}^{1000} n = \frac{(1000)(1001)}{2} = 500{,}500$

2. $\sum_{n=100}^{199} n = \frac{(100)(100+199)}{2} = 14{,}950$

3. $S = \frac{1}{1-\left(-1/3\right)} = \frac{1}{4/3} = \frac{3}{4}$

4. Since $\sin\left(180^\circ + n\right) = -\sin n$, all of the values cancel, resulting in a sum of 0.

5. There are 9 single-digit integers, followed by 90 double-digit integers, for a total of 189 digits. Therefore, to get to the 2011th digit, we must add $2011-189=1822$ more digits. Each integer from that point takes three digits, so we may use $\frac{1822}{3}=607$ more integers with a remainder of 1. $99+607=706$, so the 2011th digit is the first 7 in 707.

6. Because the terms alternate in sign and strictly decrease in absolute value, s will be larger than any partial sum of the series with an even number of terms and will be smaller than any partial sum of the series with an odd number of terms. Therefore, $\frac{7}{12} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} < s < 1-\frac{1}{2}+\frac{1}{3} = \frac{5}{6}$ (the exact value of the sum is $\ln 2 \approx 0.693$).

7. $\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{n^m} = \sum_{n=2}^{\infty}\frac{1/n^2}{1-1/n} = \sum_{n=2}^{\infty}\frac{1}{n(n-1)} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right) = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots$, so the sum is 1

8. Every four consecutive terms cancel, so $i+i^2+i^3+i^4+\ldots+i^n+\ldots+i^{2011} = i^{2009}+i^{2010}$ $+i^{2011} = i-1-i = -1$.

9. Let a and b be the first terms of the arithmetic and geometric sequences, respectively, and let d and r be the common difference and common ratio of the two sequences. Then $a+b=3$, $a+d+br=8$, and $a+2d+br^2=15$, and subtracting the first equation from the second and the second from the third yields

$d+b(r-1)=5$ and $d+b(r^2-r)=7$. Now, multiplying the first of these two equations by r and subtracting the second from that yields $dr-d=5r-7$ $\Rightarrow d=\frac{5r-7}{r-1}=5-\frac{2}{r-1}$. Since d and r are both positive integers, we must have that 2 is divisible by $r-1$, meaning $r=2$ or $r=3$. If $r=2$, then $d=3$, and if $r=3$, then $d=4$, so the sum of the possible values of d is $3+4=7$ since both values yield acceptable sequences.

10. Let x be this limit. Then $x=\frac{1}{1+x}\Rightarrow x^2+x=1\Rightarrow 0=x^2+x-1=0\Rightarrow x=\frac{-1\pm\sqrt{1+4}}{2}$ $=\frac{-1\pm\sqrt{5}}{2}$, but the continued fraction is positive, being made up of only positive numbers, so the limit is $\frac{-1+\sqrt{5}}{2}$.

11. Let $X=\sum_{i=1}^{\infty}\frac{1}{(2i-1)^2}$. Multiplying all the terms in the series $\sum_{i=1}^{\infty}\frac{1}{i^2}$ by $\frac{1}{4}$ yields the series made up of the reciprocals of the even perfect squares. Therefore, $X+\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{6}\Rightarrow X=\frac{3}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$.

12. This sequence is the same as if defined as $a_{n+1}-a_n=\frac{a_n-a_{n-1}}{2}$ with $a_1-a_0=1$, it is easy to see that $a_{n+1}-a_n=\left(\frac{1}{2}\right)^n$, and since $a_{n+1}=(a_{n+1}-a_n)+(a_n-a_{n-1})+\dots+(a_1-a_0)$ $+a_0$, we can see that $a_{n+1}=\sum_{i=0}^{n}\left(\frac{1}{2}\right)^i$. As n approaches ∞, a_n approaches $\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i$ $=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2$.

13. $i\log 1+i^2\log 2+\dots+i^n\log n+\dots+i^{10}\log 10=(-\log 2+\log 4-\log 6+\log 8-\log 10)$ $+(\log 1-\log 3+\log 5-\log 7+\log 9)i=\log\left(\frac{4}{15}\right)+i\log\left(\frac{15}{7}\right)$

14. $S_{2011}=\sum_{i=1}^{2011}\left((-1)^{i+1}i\right)=(1-2)+(3-4)+\dots+(2009-2010)+2011=1005(-1)+2011$ $=2011-1005=1006$

15. $\prod_{n=2}^{2011}\left(\sum_{i=0}^{\infty}\frac{1}{n^i}\right)=\prod_{n=2}^{2011}\frac{1}{1-\frac{1}{n}}=\prod_{n=2}^{2011}\frac{n}{n-1}=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{2011}{2010}=2011$

16. Since there are exactly 2007 integers in the partition, each must be at least 1, giving a total of 4 to be spread among the 2007 integer copies of 1. This would be the same as the number of partitions of 4, which is given as 5 in the problem. Therefore, the sought number of partitions is also 5.

17. Since $13-11=2$ and $11-5=6$, and because the sequence is arithmetic, 11 must be three times as many terms away from 5 in the sequence as it is from 13. The only possibility is if the sequence begins 5, 7, 9, 11, 13, …, and the 20th term of this sequence is $a_{20}=5+(20-1)(2)=5+19\cdot 2=5+38=43$.

18. The last person must have passed the exam, so we must distribute the other two passed exams among the 7 people. Therefore, the number of sequences is $\binom{7}{2}=21$.

19. $5400=2^3 3^3 5^2$, so the divisors that work to make the function value non-zero are 2^1, 2^2, 2^3, 3^1, 3^2, 3^3, 5^1, and 5^2, and the function values for these numbers are $\ln 2$ (three times), $\ln 3$ (three times), and $\ln 5$ (two times). Therefore, this series is equal to $3\ln 2+3\ln 3+2\ln 5=\ln\left(2^3 3^3 5^2\right)=\ln 5400$.

20. A could be the same since it would match in the nth term. C could be the same since it would match in the 1st term. B would be different because it would differ from the nth sequence in their nth terms. Thus D is wrong also. This is similar to Cantor's diagonal argument for the uncountability of the real numbers.

21. Let S be the sum of the series, so $S=\frac{3}{3}+\frac{5}{9}+\frac{7}{27}+\frac{9}{81}+\ldots$. Multiplying this series by $\frac{1}{3}$ and subtracting from the original yields $\frac{2}{3}S=\frac{3}{3}+\left(\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\ldots\right)=1+\frac{\frac{2}{9}}{1-\frac{1}{3}}$

$=1+\frac{\frac{2}{9}}{\frac{2}{3}}=1+\frac{1}{3}=\frac{4}{3}\Rightarrow S=2$.

22. Plugging in the term numbers gives $a+b+c=1$, $4a+2b+c=3$, and $9a+3b+c=1$. Subtracting the first equation from the second and the second from the third yields $3a+b=2$ and $5a+b=-2$. Subtracting the first of these equations from the second

yields $2a=-4 \Rightarrow a=-2 \Rightarrow b=8 \Rightarrow c=-5$, so the terms in the sequence are defined by $-2n^2+8n-5$. Therefore, the fourth term would be $-2(4)^2+8(4)-5=-5$.

23. Counting to n one time through takes n numbers, so it would take Aaya $1+2+3+\ldots$ $+100=\frac{100\cdot 101}{2}=5050$.

24. The pattern is 1, 1, 2, 4, 8, 16, 32, …, 2^{n-2}, …, so the 2011th term is 2^{2009}.

25. Since 2011 leaves a remainder of 1 when divided by 3 and it is odd, the next term needs to leave a remainder of 2 when divided by 3 and be even, so it must be of the form $6n+2$ for some nonnegative n. The next term would have to be similar to the first term, so it must be of the form $6n+1$ for some nonnegative n. The first four positive terms not of either of these forms are 3, 4, 5, and 6, and their sum is 18.

26. The net displacement the mouse moves in the "ahead" direction is $4-1+\frac{1}{4}-\frac{1}{16}+\ldots$ $=\frac{4}{1-\left(-1/4\right)}=\frac{4}{5/4}=\frac{16}{5}$, and the net displacement the mouse moves in the perpendicular direction is $2-\frac{1}{2}+\frac{1}{8}-\frac{1}{32}+\ldots=\frac{2}{1-\left(-1/4\right)}=\frac{2}{5/4}=\frac{8}{5}$, so the total distance the mouse is away from its starting point is $\sqrt{\left(8/5\right)^2+\left(16/5\right)^2}$ $=\sqrt{64/25+256/25}=\sqrt{320/25}=\frac{8\sqrt{5}}{5}$.

27. $1+(1-i)+(1-i)^2+\ldots+(1-i)^{n-1}+\ldots+(1-i)^{10}=\frac{1\left(1-(1-i)^{11}\right)}{i}=\frac{1-(-2i)^5(1-i)}{i}$ $=\frac{1+32i(1-i)}{i}=\frac{33+32i}{i}=32-33i$

28. $27=\frac{a}{r}\cdot a\cdot ar=a^3 \Rightarrow a=3$ since the sequence consisted of real terms

29. The numbers in the series become so small that the series can be approximated by the sum of the infinite geometric series. Therefore, the sum is approximately $\frac{1}{1-21/41}=\frac{1}{20/41}=\frac{41}{20}=2.050$.

30. First, $\left|z_n\right| = \dfrac{13}{2^n}$, so to land inside the unit circle, n must be at least 4. Now, since $\dfrac{i}{2} = \dfrac{1}{2}\text{c}is90^\circ$ and $5+12i$ is in the first quadrant, we must rotate it through 270° to get the new complex number into the fourth quadrant. Therefore, it would take 3 more 90° rotations past the first four, which put the complex number back in the first quadrant. This is a total of 7 rotations, so $n=7$ is the smallest such value of n.