MU ALPHA THETA

NATIONAL CONVENTION
SAN DIEGO 2013

Complex Numbers
Alpha, Round 1
Test #123

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select “E) NOTA” and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select “E) NOTA” in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct. Furthermore, assume that $i=\sqrt{-1}$, $\operatorname{cis}\theta=\cos\theta+i\sin\theta$, and $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ are the real and imaginary parts of z, respectively, unless otherwise specified.

Assume all answer choices have the correct units.

Good luck!

1. Compute $\sqrt{-1}\cdot\sqrt{-3}\cdot\sqrt{-11}\cdot\sqrt{-61}$.

   (A) $-i\sqrt{2013}$ (B) $-\sqrt{2013}$ (C) $\sqrt{2013}$ (D) $i\sqrt{2013}$ (E) NOTA

2. Compute i^{2013}.

   (A) i (B) -1 (C) $-i$ (D) 1 (E) NOTA

3. Compute $\left|\frac{3+4i}{5+12i}\right|$.

   (A) $\frac{5}{13}$ (B) $\frac{25}{169}$ (C) $\frac{13}{5}$ (D) $\frac{169}{25}$ (E) NOTA

4. Compute $\sqrt{\frac{i}{4}+\sqrt{\frac{i}{4}+\sqrt{\frac{i}{4}+\cdots}}}$.

   (A) $\frac{1}{2}+2^{-3/4}\operatorname{cis}\left(\frac{\pi}{8}+\pi k\right), k=0,1$ (B) $\frac{1}{2}+2^{-1/4}\operatorname{cis}\left(\frac{\pi}{8}+2\pi k\right), k=0,1$

   (C) $\frac{1}{2}+2^{-1/2}\operatorname{cis}\left(\frac{\pi}{8}+\pi k\right), k=0,1$ (D) $\frac{1}{2}+2^{1/4}\operatorname{cis}\left(\frac{\pi}{8}+\pi k\right), k=0,1$ (E) NOTA

5. Compute $\dfrac{\prod_{n=1}^{45}\operatorname{Re}\left[\operatorname{cis}((2n-1)^\circ)\right]}{\prod_{n=1}^{45}\operatorname{Im}\left[\operatorname{cis}(2(2n-1)^\circ)\right]}$.

   (A) 2^{-1} (B) $2^{-1/2}$ (C) $2^{1/2}$ (D) 2 (E) NOTA

6. Let $a = i^{k_1} + i^{k_2} + i^{k_3} + i^{k_4}$, where each k_i is randomly chosen from the set $\{1,2,3,4\}$. What is the probability that $a = 0$?

(A) $\frac{7}{64}$ (B) $\frac{9}{64}$ (C) $\frac{37}{256}$ (D) $\frac{39}{256}$ (E) NOTA

7. The solutions to the equation $z^6 = 729$ can be written in the form $z_k = 3(\cos\theta_k + i\sin\theta_k)$ where $k \in \{1,2,3,4,5,6\}$ and $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 < \theta_6 < 2\pi$. Compute the value of $|z_3 - z_6|$.

(A) $\frac{3\sqrt{3}}{2}$ (B) 3 (C) $3\sqrt{3}$ (D) 6 (E) NOTA

8. Consider two complex numbers $z = a + bi$ and $w = c + di$, $a,b,c,d \in \mathbb{R}$, as well as the vectors $v_1 = \langle \text{Re}(z), \text{Im}(z) \rangle$ and $v_2 = \langle \text{Re}(w), \text{Im}(w) \rangle$. Which of the following is equal to $v_1 \cdot v_2$?

(A) $\text{Re}(z \cdot w)$ (B) $\text{Re}(\bar{z} \cdot \bar{w})$ (C) $\text{Re}(\overline{zw})$ (D) $\text{Re}(\bar{z} \cdot w)$ (E) NOTA

9. The binomial coefficient, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, can be written as

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

in order to accommodate for all complex n. Using this, compute $\binom{i}{4}$.

(A) $-\frac{5}{12}$ (B) $-\frac{9}{24}$ (C) $\frac{9}{24}$ (D) $\frac{5}{12}$ (E) NOTA

10. Define the operation $\otimes$ as $\text{cis}(\alpha) \otimes \text{cis}(\beta) = \cos\beta \, \text{cis}\,\alpha$. There exist real numbers $0^\circ < \theta < 90^\circ$ and $0^\circ < \varphi < 90^\circ$ (both in degrees) such that $\text{cis}\theta + \text{cis}\varphi = 2(\text{cis}43^\circ \otimes \text{cis}35^\circ)$. Compute $\theta\varphi$, ignoring units in your answer.

(A) 78 (B) 624 (C) 1248 (D) 1505 (E) NOTA

11. Region R is bounded by the set of all complex numbers on the Argand plane $z = a + bi$ where a and b are real numbers satisfying the equation $a^2 - 12a - 13 = 6 + 6b - b^2$. Compute the area of R.

(A) 16π (B) 32π (C) 64π (D) 128π (E) NOTA

12. A complex number z satisfies the equation $\left|z - |z|\right| = \sqrt{2}$. Given that $\text{Re}(z) = a$, determine the value of $|z|$ in terms of a.

(A) $\dfrac{1}{1-a}$ (B) $\dfrac{a - \sqrt{a^2+1}}{2}$ (C) $\dfrac{a + \sqrt{a^2+1}}{2}$ (D) $\dfrac{a + \sqrt{a^2+4}}{2}$ (E) NOTA

13. Consider two vectors $v_1 = \langle x, y \rangle$ and $v_2 = \langle 1+i, 3+2i \rangle$, where $x, y \in \mathbb{R}$. Given that $v_1 \cdot v_2 = 5 + 6i$, compute $x + y$.

(A) 3 (B) 4 (C) 6 (D) 7 (E) NOTA

14. Consider the matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$. Determine all possible values of λ such that the determinant of $B = A - \lambda I$ is 0, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(A) $1 \pm 2i$ (B) $2 \pm 3i$ (C) $3 \pm 4i$ (D) $5 \pm 6i$ (E) NOTA

15. The fourth roots of unity are the solutions to the equation $x^4 - 1 = 0$. Graphing these roots on the complex plane gives a square which we call S_1. Form a sequence of squares S_i, where the nth square S_n is formed by connecting the midpoints of S_{n-1}. For example, by connecting the midpoints of S_1, we obtain S_2. Now, let $f_n(x)$ be the polynomial with roots equal to the vertices of S_n. Compute $\sum_{n=1}^{\infty} f_n(0)$.

(A) $-\dfrac{4}{5}$ (B) $-\dfrac{1}{3}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{4}{5}$ (E) NOTA

16. Consider Eli's function, $f(x)=9x^2+5x+k^2$, where x and k are real. Determine the interval for k such that the graph of $f(x)$ does not intersect the x-axis.

(A) $\left(-\infty,-\frac{5}{6}\right)$ (B) $\left(\frac{5}{6},\infty\right)$ (C) $\left(-\frac{5}{6},\frac{5}{6}\right)$ (D) $\left(-\infty,-\frac{5}{6}\right)\cup\left(\frac{5}{6},\infty\right)$ (E) NOTA

17. Consider Patrick's function, $f(x)=(x-1)\left(1+\sum_{n=0}^{2012}nx^n\right)$. f has 2013 roots; one of which is equal to 1, while the other 2012 are imaginary. Let S_2 equal the sum of the products of the roots of $f(x)$ taken two at a time. Compute S_2.

(A) $-\frac{4021}{2012}$ (B) $-\frac{1}{2012}$ (C) $\frac{1}{2012}$ (D) $\frac{4021}{2012}$ (E) NOTA

18. Consider Laura's imaginary, six-sided, fair dice. The sides on the first die show the numbers $i, 2i, 3i, 4, 5, 6,$ while the sides of the second die show the numbers $1, 2, 3, 4i, 5i, 6i.$ Laura takes rolls both the dice once and multiplies the numbers shown. If the result is real, the player gets the absolute value of the result in dollars. If the result is imaginary, the player loses the absolute value of the result in dollars. What is the expected value of this game, in dollars?

(A) –\$2.25 (B) –\$1.25 (C) \$1.25 (D) \$2.25 (E) NOTA

19. Consider Caleb's favorite expression, $C=\left(1+i\sqrt{3}\right)^{2013}$. Evaluate C.

(A) -2^{2013} (B) 2^{2013} (C) $2^{2012}\left(1+i\sqrt{3}\right)$ (D) $2^{2012}\left(-1+i\sqrt{3}\right)$ (E) NOTA

20. Consider Will's geometric series. He doesn't care what the first term is, as long as it's not zero. He would, however, like a common ratio such that at some point in the geometric series, the nth term equals the first term. What is the smallest value of n such that there are exactly 50 possible common ratios for the series located in the second quadrant of the Argand plane? Do not consider points on the axes.

(A) 200 (B) 201 (C) 202 (D) 203 (E) NOTA

21. The cosine function can be approximated by the function $f(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}$. Evaluate $f(i)$.

(A) $\frac{11}{24}$ (B) $\frac{13}{24}$ (C) 1 (D) $\frac{37}{24}$ (E) NOTA

22. Calculate $\prod_{\theta=1}^{2013} \operatorname{cis}\theta°$.

(A) $\operatorname{cis}(1006\cdot 2013)°$ (B) $\operatorname{cis}(1007\cdot 2013)°$

(B) $\operatorname{cis}(2012\cdot 2013)°$ (D) $\operatorname{cis}(2013\cdot 2014)°$ (E) NOTA

23. A sequence of points $z_k=\sqrt{\binom{k+1}{2}}\,i^k$ is plotted on the Argand plane. A bug begins at z_1 and travels along the segments $z_1z_2, z_2z_3, \ldots, z_nz_{n+1}, \ldots, z_{2012}z_{2013}$. Let D be the total distance the bug travels. Find the remainder when D is divided by 100.

(A) 18 (B) 19 (C) 90 (D) 91 (E) NOTA

24. Consider the matrix $A=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$, and the complex vector, $z=\begin{pmatrix}3\\4i\end{pmatrix}$. Calculate $A^{37}z$.

(A) $\frac{1}{2}\begin{pmatrix}-3-4i\sqrt{3}\\ -3\sqrt{3}-4i\end{pmatrix}$ (B) $\frac{1}{2}\begin{pmatrix}3-4i\sqrt{3}\\ 3\sqrt{3}+4i\end{pmatrix}$ (C) $\frac{1}{2}\begin{pmatrix}-3+4i\sqrt{3}\\ 3\sqrt{3}-4i\end{pmatrix}$ (D) $\frac{1}{2}\begin{pmatrix}3+4i\sqrt{3}\\ 3\sqrt{3}+4i\end{pmatrix}$ (E) NOTA

25. Given that $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$, compute the value of $\sum_{k=0}^{\infty}\left(\frac{i^k}{\left(\sum_{n=1}^{k} n(n!)\right)+1}\right)$.

(A) $-ie^i-i$ (B) ie^i-i (C) $-ie^i+i$ (D) ie^i+i (E) NOTA

26. Let $f(x)=x^{2013}+x^{2012}+\cdots+x+1$. Denote $R(n)$ as the remainder when $f(x)$ is divided by $(x-n)$. Compute the remainder when $\sum_{k=0}^{2013} R(i^k)$ is divided by 1000.

(A) 0 (B) 42 (C) 52 (D) 56 (E) NOTA

27. Consider a sequence of functions $f_n(x)=\sum_{j=0}^{2^n-1} x^j$. Let the set ψ contain all arguments $0\le\theta<2\pi$ such that $\text{cis}\,\theta$ is a solution to $f_n(x)=0$ for a given n. Determine the smallest positive integer n such that the sum of the entries in ψ is greater than or equal to 2013π.

(A) 10 (B) 11 (C) 12 (D) 13 (E) NOTA

28. A *Gaussian Integer* is a complex number $z=a+bi$ where $a,b\in\mathbb{Z}$. Consider the Gaussian Integer $z=m+3ni$. Which of the following is not a possible value for $|z|^2$?

(A) 2010 (B) 2011 (C) 2012 (D) 2013 (E) NOTA

Use the following information for Problems 29 and 30:

In Problem 28, we defined the Gaussian Integers. Another set of integers, called the *Eisenstein Integers*, are defined, for a,b integers, as

$$z=a+b\omega; \qquad \omega=\frac{1}{2}\left(-1+i\sqrt{3}\right)=e^{2\pi i/3}.$$

Because the argument of ω is $\frac{2\pi}{3}$, or $60°$, the Eisenstein Integers form a triangular lattice on the Argand Plane, whereas the Gaussian Integers form a square lattice.

29. The triangular region T has vertices located at the points $z_1=-\omega$, $z_2=3+2\omega$, and $z_3=2\omega$. Compute the area of T. *(Hint: Graph the points with shifted axes – by what angle would you shift them by?).*

(A) $\frac{\sqrt{3}}{4}$ (B) $\sqrt{3}$ (C) $\frac{9\sqrt{3}}{4}$ (D) $4\sqrt{3}$ (E) NOTA

30. The absolute value of the Gaussian Integers is simply $|z| = \sqrt{a^2 + b^2}$. This is not the case for the Eisenstein Integers. For the general Eisenstein Integer $z = a + b\omega$, deduce the absolute value in terms of a and b.

(A) $\sqrt{(a-b)^2 - ab}$ (B) $\sqrt{(a-b)^2}$ (C) $\sqrt{(a-b)^2 + ab}$

(D) $\sqrt{(a+b)^2}$ (E) NOTA